

# (Co)associative 3-ary (co)algebras and infinitesimal bialgebras: construction and main properties

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**ABSTRACT.** The (co)associative, partially (co)associative and totally (co)associative 3-ary (co)algebras and infinitesimal bialgebras are constructed and discussed. Their trimodules and matched pairs are defined and completely characterized. The main structural properties and relations are also deduced and analyzed.

**Keywords.** Associative 3-ary algebra, coassociative 3-ary coalgebra and associative 3-ary bialgebra.

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## 1. Introduction

An  $n$ -ary algebra is a linear space endowed with an internal composition law involving  $n$  elements:  $\mu : V^{\otimes n} \rightarrow V$ . These  $n$ -ary algebras,  $n \geq 3$ , knew a versatile development since the discovery of the Nambu mechanics in 1973 (see [31]) and the work by S. Okubo [32] on Yang-Baxter equation. The  $n$ -ary products were also defined by cubic matrices and a generalization of the notion of determinant, called hyperdeterminant, first introduced by Cayley in 1840, then rediscovered and generalized by Sokolov in 1972 [35], and, still later, by Kapranov, Gelfand and Zelevinskii in 1994 [21].

The ternary algebraic structures are particularly of a great potential application in various domains of physics and mathematics, and data processing. Their subclass known as Bagger-Lambert algebras [6] are involved in string theory and  $M$ -branes. A good compilation of their applications can be found in the work by Kerner (see [21]–[24]) on ternary and non-associative structures and their applications in physics. This author investigated the use of  $Z_3$ -graded structures instead of  $Z_2$ -graded in physics, leading to interesting results in the construction of gauge theories.

The  $n$ -ary algebras of associative type were studied by Lister, Loos, Myung and Carlsson (see [11], [12], [27], [28]). Relatively to their structure, these algebras encompass two main classes: totally associative  $n$ -ary algebras and partially associative  $n$ -ary algebras, which also generate other interesting variants of algebras.

**DEFINITION 1.1.** A totally associative 3-ary algebra is a  $K$ -vector space  $\mathcal{T}$  endowed with a trilinear operation  $\mu$  satisfying, for all  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$ :

$$\begin{aligned} \mu(\mu(x_1 \otimes x_2 \otimes x_3) \otimes x_4 \otimes x_5) &= \mu(x_1 \otimes \mu(x_2 \otimes x_3 \otimes x_4) \otimes x_5) \\ &= \mu(x_1 \otimes x_2 \otimes \mu(x_3 \otimes x_4 \otimes x_5)). \end{aligned} \quad (1.1)$$

**EXAMPLE 1.2.** Let  $\mathcal{T}$  be a 2-dimensional space vector with a basis  $\{e_1, e_2\}$ . The trilinear product  $\mu$  on  $\mathcal{T}$  defined by

$$\begin{aligned} \mu(e_1 \otimes e_1 \otimes e_1) &= e_1 & \mu(e_2 \otimes e_2 \otimes e_1) &= e_1 + e_2 \\ \mu(e_1 \otimes e_1 \otimes e_2) &= e_2 & \mu(e_2 \otimes e_2 \otimes e_2) &= e_1 + 2e_2 \end{aligned}$$

$$\begin{aligned}\mu(e_1 \otimes e_2 \otimes e_1) &= e_2 & \mu(e_1 \otimes e_2 \otimes e_2) &= e_1 + e_2 \\ \mu(e_2 \otimes e_1 \otimes e_1) &= e_2 & \mu(e_2 \otimes e_1 \otimes e_2) &= e_1 + e_2\end{aligned}$$

defines a totally associative 3-ary algebra.

DEFINITION 1.3. A weak totally associative 3-ary algebra is a  $\mathcal{K}$ -vector space  $\mathcal{W}$  equipped with a trilinear operation  $\mu$  satisfying, for all  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{W}$  :

$$\mu(\mu(x_1 \otimes x_2 \otimes x_3) \otimes x_4 \otimes x_5) = \mu(x_1 \otimes x_2 \otimes \mu(x_3 \otimes x_4 \otimes x_5)). \quad (1.2)$$

REMARK 1.4. Naturally, any totally associative 3-ary algebra is a weak totally associative 3-ary algebra.

DEFINITION 1.5. A partially associative 3-ary algebra is a  $\mathcal{K}$ -vector space  $\mathcal{P}$  with a trilinear operation  $\mu$  satisfying, for all  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{P}$  :

$$\begin{aligned}\mu(\mu(x_1 \otimes x_2 \otimes x_3) \otimes x_4 \otimes x_5) + \mu(x_1 \otimes \mu(x_2 \otimes x_3 \otimes x_4) \otimes x_5) + \\ \mu(x_1 \otimes x_2 \otimes \mu(x_3 \otimes x_4 \otimes x_5)) = 0.\end{aligned} \quad (1.3)$$

EXAMPLE 1.6. Let  $\mathcal{P}$  be a 2-dimensional vector space with a basis  $\{e_1, e_2\}$ . The trilinear product  $\mu$  on  $\mathcal{P}$  determined by

$$\mu(e_1 \otimes e_1 \otimes e_1) = e_2; \quad \mu(e_i \otimes e_j \otimes e_k) = 0, \quad i, j, k = 1, 2 \text{ with } (i, j, k) \neq (1, 1, 1).$$

defines a partially associative 3-ary algebra.

REMARK 1.7. Let  $(\mathcal{A}, m)$  be a bilinear associative algebra. Then, the trilinear operation defined on  $\mathcal{A}$  by

$$\mu(x_1 \otimes x_2 \otimes x_3) = m(m(x_1 \otimes x_2) \otimes x_3)$$

produces on the vector space  $\mathcal{A}$  a structure of totally associative 3-ary algebra which is not partially associative.

From the Remark 1.7, we can deduce a series of examples of totally associative 3-ary algebras, and hence of weak totally associative 3-ary algebras. In the next example, we provide a class of compatible 2-dimensional totally associative 3-ary algebras derived from 2-dimensional associative algebras.

EXAMPLE 1.8. Let  $(\mathcal{A}, m)$  be an associative algebra with a basis  $\{e_1, e_2\}$ . We denote by  $(\mathcal{A}, \mu)$ ,  $i = 1, 7$ , the related compatible totally associative 3-ary algebras defined as follows:

- $\mathcal{A}_1$ :  $m(e_1 \otimes e_1) = e_1, m(e_2 \otimes e_2) = e_2; \quad \mu(e_1 \otimes e_1 \otimes e_1) = e_1, \mu(e_2 \otimes e_2 \otimes e_2) = e_2.$
- $\mathcal{A}_2$ :  $m(e_2 \otimes e_2) = e_2, m(e_1 \otimes e_2) = m(e_2 \otimes e_1) = e_1; \quad \mu(e_2 \otimes e_2 \otimes e_2) = e_2, \mu(e_1 \otimes e_2 \otimes e_2) = \mu(e_2 \otimes e_1 \otimes e_2) = \mu(e_2 \otimes e_2 \otimes e_1) = e_1.$
- $\mathcal{A}_3$ :  $m(e_1 \otimes e_1) = e_1; \quad \mu(e_1 \otimes e_1 \otimes e_1) = e_1.$
- $\mathcal{A}_4$ :  $m(e_i \otimes e_j) = 0, i, j = 1, 2; \quad \mu(e_i \otimes e_j \otimes e_k) = 0, i, j, k = 1, 2.$
- $\mathcal{A}_5$ :  $m(e_1 \otimes e_1) = e_2; \quad \mu(e_i \otimes e_j \otimes e_k) = 0, i, j, k = 1, 2.$
- $\mathcal{A}_6$ :  $m(e_2 \otimes e_1) = e_1, m(e_2 \otimes e_2) = e_2; \quad \mu(e_2 \otimes e_2 \otimes e_1) = e_1, \mu(e_2 \otimes e_2 \otimes e_2) = e_2.$
- $\mathcal{A}_7$ :  $m(e_1 \otimes e_2) = e_1, m(e_2 \otimes e_2) = e_2; \quad \mu(e_1 \otimes e_2 \otimes e_2) = e_1, \mu(e_2 \otimes e_2 \otimes e_2) = e_2.$

Note that the 3-ary algebras given by subspaces of an associative algebra, closed under the ternary product  $(x, y, z) \mapsto xyz$ , are linked to the ternary operation defined by Hestenes [18] on a linear space of rectangular matrices  $A, B, C \in \mathcal{M}_{m,n}$ , with complex entries by  $AB^*C$ , where  $B^*$  is the conjugate transpose matrix of  $B$ . This operation, strictly speaking, does not define a ternary algebra product on  $\mathcal{M}_{m,n}$  as it is linear on the first and the third arguments, but conjugate-linear on the second argument. It satisfies identities, sometimes referred to as identities of total associativity of second kind, which only slightly differ from the identities of totally associative algebras.

The totally associative ternary algebras are also sometimes called associative triple systems.

The cohomology of totally associative  $n$ -ary algebras was studied by Carlsson through the embedding [12]. In [3], the 1-parameter formal deformation theory was extended to ternary

algebras of associative type, while in [4] discussions were made on their cohomologies in the context of deformations. See also [19], [16], and [34] (and references therein).

The extension of the notion of associativity to  $n$ -ary product is not trivial. The most natural procedure might be based on the notion of totally associativity. Unfortunately, this notion is not auto-dual in the operadic point of view. So, it is necessary to introduce a most general notion of associativity under the concept of partial associativity.

In this paper, we focus on the case  $n = 3$ . Of course, the structure of classes of associative  $n$ -ary algebras, (for  $n \geq 3$ ), is more complicated than that of associative algebras. Hence, their exhaustive investigation in order to derive their relevant relationships is of some importance in algebra. Besides, the concepts of associative coalgebras and bialgebras are fundamental in the theory of associative algebras. In [1] and [2], Aguiar developed the basic theory of infinitesimal bialgebras and infinitesimal hopf bialgebras. An infinitesimal bialgebra is at the same time an algebra and a coalgebra, in such a way that the comultiplication is a derivation. Aguiar established many properties of ordinary Hopf algebras which possess infinitesimal version. He introduced bicrossproducts, quasitriangular infinitesimal bialgebras, the corresponding infinitesimal Yang-Baxter equation and a notion of Drinfeld's double for infinitesimal Hopf algebras. He also showed that non degenerate antisymmetric solutions of associative Yang-Baxter equations are in one-to-one correspondence with non degenerate cyclic 2-cocycles. Furthermore, Bai established a clear analogy between the antisymmetric infinitesimal bialgebras and the dendriform  $D$ -bialgebras [7]. Motivated by all these studies, we define, in this work, the concepts of totally and partially coassociative 3-ary coalgebras and totally and partially associative 3-ary infinitesimal bialgebras, and investigate their main properties as well as their relationships with associative 3-ary infinitesimal bialgebras.

The paper is organized as follows. In section 2, we construct partially and totally coassociative 3-ary coalgebras and discuss their main properties. Section 3 is devoted to the construction of trimodules and matched pairs of totally and partially associative 3-ary algebras. Then, in Section 4, we define the partially and totally associative 3-ary infinitesimal bialgebras and investigate their relation with associative 3-ary algebras. Section 5 is devoted to concluding remarks.

Throughout the paper, we consider a fixed field  $\mathcal{K}$  of characteristic zero.

## 2. Coassociative 3-ary coalgebras

In this section, we introduce and develop the concepts of totally coassociative 3-ary coalgebra, weak totally coassociative 3-ary coalgebra and partially coassociative 3-ary coalgebra.

**2.1. Definitions.** Let us start with the following definitions.

**DEFINITION 2.1.** *A totally associative 3-ary algebra is a  $\mathcal{K}$ -vector space  $\mathcal{T}$  equipped with a trilinear operation  $\mu$  satisfying*

$$\mu \circ (\mu \otimes \text{id} \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \text{id} \otimes \mu). \quad (2.1)$$

**DEFINITION 2.2.** *A weak totally associative 3-ary algebra is a  $\mathcal{K}$ -vector space  $\mathcal{W}$  with a trilinear operation  $\mu$  satisfying*

$$\mu \circ (\mu \otimes \text{id} \otimes \text{id}) = \mu \circ (\text{id} \otimes \text{id} \otimes \mu). \quad (2.2)$$

**DEFINITION 2.3.** *A partially associative 3-ary algebra is a  $\mathcal{K}$ -vector space  $\mathcal{P}$  endowed with a trilinear operation  $\mu$  satisfying*

$$\mu \circ (\mu \otimes \text{id} \otimes \text{id} + \text{id} \otimes \mu \otimes \text{id} + \text{id} \otimes \text{id} \otimes \mu) = 0. \quad (2.3)$$

Let  $(\mathcal{A}, \mu)$  be an associative 3-ary algebra and  $\mathcal{A}^*$  be its dual space. Then, we get the dual mapping  $\mu^* : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$  of  $\mu$ , for every  $x, y, z \in \mathcal{A}$  and  $\xi, \eta, \gamma \in \mathcal{A}^*$ ,

$$\langle \mu^*(\xi), x \otimes y \otimes z \rangle = \langle \xi, \mu(x, y, z) \rangle, \quad (2.4)$$

$$\langle \xi \otimes \eta \otimes \gamma, x \otimes y \otimes z \rangle = \langle \xi, x \rangle \langle \eta, y \rangle \langle \gamma, z \rangle, \quad (2.5)$$

where  $\langle, \rangle$  is the natural nondegenerate symmetric bilinear form on the vector space  $\mathcal{A} \oplus \mathcal{A}^*$  defined by  $\langle \xi, x \rangle = \xi(x)$ ,  $\xi \in \mathcal{A}^*$ ,  $x \in \mathcal{A}$ .

Further, by the definition of associative 3-ary algebras, we have  $Im(\mu^*) \subseteq \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$ , and

$$(\mu^* \otimes id \otimes id) \circ \mu^* = (id \otimes \mu^* \otimes id) \circ \mu^* = (id \otimes id \otimes \mu^*) \circ \mu^*$$

and

$$(\mu^* \otimes id \otimes id + id \otimes \mu^* \otimes id + id \otimes id \otimes \mu^*) \circ \mu^* = 0,$$

for the totally associative 3-ary algebras and the partially associative 3-ary algebras, respectively. That is, for every  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$  and  $\xi \in \mathcal{A}^*$ , we have

$$\begin{aligned} & \langle (\mu^* \otimes id \otimes id) \circ \mu^*(\xi), x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \rangle \\ &= \langle \xi, \mu \circ (\mu \otimes id \otimes id)(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) \rangle \\ &= \langle \xi, \mu(\mu(x_1 \otimes x_2 \otimes x_3) \otimes x_4 \otimes x_5) \rangle \\ &= \langle \xi, \mu(x_1 \otimes \mu(x_2 \otimes x_3 \otimes x_4) \otimes x_5) \rangle \\ &= \langle (id \otimes \mu^* \otimes id) \circ \mu^*(\xi), x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \rangle \\ &= \langle \xi, \mu(x_1 \otimes x_2 \otimes \mu(x_3 \otimes x_4 \otimes x_5)) \rangle \\ &= \langle (id \otimes id \otimes \mu^*) \circ \mu^*(\xi), x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \rangle \end{aligned}$$

and

$$\begin{aligned} & \langle (\mu^* \otimes id \otimes id + id \otimes \mu^* \otimes id + id \otimes id \otimes \mu^*) \circ \mu^*(\xi), x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \rangle \\ &= \langle \xi, \mu \circ (\mu \otimes id \otimes id + id \otimes \mu \otimes id + id \otimes id \otimes \mu)(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) \rangle, \end{aligned}$$

respectively.

Provided the above, the following definitions are in order.

**DEFINITION 2.4.** A totally coassociative 3-ary coalgebra  $(\mathcal{T}, \Delta)$  is a vector space  $\mathcal{T}$  with a linear mapping  $\Delta : \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T}$  satisfying

$$(\Delta \otimes id \otimes id) \circ \Delta = (id \otimes \Delta \otimes id) \circ \Delta = (id \otimes id \otimes \Delta) \circ \Delta. \quad (2.6)$$

The Eq. 2.6 is also called standard ternary coassociativity condition in [13].

**DEFINITION 2.5.** A weak totally coassociative 3-ary coalgebra  $(\mathcal{W}, \Delta)$  is a vector space  $\mathcal{W}$  with a linear mapping  $\Delta : \mathcal{W} \rightarrow \mathcal{W} \otimes \mathcal{W} \otimes \mathcal{W}$  satisfying

$$(\Delta \otimes id \otimes id) \circ \Delta = (id \otimes id \otimes \Delta) \circ \Delta. \quad (2.7)$$

**DEFINITION 2.6.** A partially coassociative 3-ary coalgebra  $(\mathcal{P}, \Delta)$  is a vector space  $\mathcal{P}$  with a linear mapping  $\Delta : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P}$  satisfying

$$(\Delta \otimes id \otimes id + id \otimes \Delta \otimes id + id \otimes id \otimes \Delta) \circ \Delta = 0. \quad (2.8)$$

**2.2. Main results.** Let us now examine the partially coassociative 3-ary coalgebras in terms of structure constants. For that, let  $(\mathcal{A}, \Delta)$  be a partially coassociative 3-ary coalgebra with a basis  $e_1, \dots, e_n$ . Assume that

$$\Delta(e_l) = \sum_{1 \leq r, s, t \leq n} c_{rst}^l e_r \otimes e_s \otimes e_t, \quad c_{rst}^l \in \mathcal{K}, 1 \leq l \leq n. \quad (2.9)$$

Then, we obtain

$$\begin{aligned} & (\Delta \otimes id \otimes id + id \otimes \Delta \otimes id + id \otimes id \otimes \Delta) \circ \Delta(e_l) \\ &= (\Delta \otimes id \otimes id + id \otimes \Delta \otimes id + id \otimes id \otimes \Delta) \left( \sum_{1 \leq r, s, t \leq n} c_{rst}^l e_r \otimes e_s \otimes e_t \right) \\ &= \sum_{1 \leq r, s, t \leq n} c_{rst}^l (\Delta(e_r) \otimes e_s \otimes e_t + e_r \otimes \Delta(e_s) \otimes e_t + e_r \otimes e_s \otimes \Delta(e_t)) \\ &= \sum_{1 \leq r, s, t \leq n} \sum_{1 \leq i, j, k \leq n} c_{rst}^l c_{ijk}^r e_i \otimes e_j \otimes e_k \otimes e_s \otimes e_t + \end{aligned}$$

$$\begin{aligned}
& \sum_{1 \leq r, s, t \leq n} \sum_{1 \leq i, j, k \leq n} c_{rst}^l c_{ijk}^s e_r \otimes e_i \otimes e_j \otimes e_k \otimes e_t + \\
& \sum_{1 \leq r, s, t \leq n} \sum_{1 \leq i, j, k \leq n} c_{rst}^l c_{ijk}^t e_r \otimes e_s \otimes e_i \otimes e_j \otimes e_k \\
& = \sum_{1 \leq r, s, t \leq n} \sum_{1 \leq i, j, k \leq n} (c_{rst}^l c_{ijk}^r + c_{irt}^l c_{jks}^r + c_{ijr}^l c_{kst}^r) e_i \otimes e_j \otimes e_k \otimes e_s \otimes e_t
\end{aligned}$$

leading to

$$\sum_{r=1}^n (c_{rst}^l c_{ijk}^r + c_{irt}^l c_{jks}^r + c_{ijr}^l c_{kst}^r) = 0, \quad 1 \leq i, j, k, s, t, l \leq n. \quad (2.10)$$

By a similar discussion, for a totally coassociative 3-ary coalgebra, we get

$$\sum_{r=1}^n c_{rst}^l c_{ijk}^r = \sum_{r=1}^n c_{irt}^l c_{jks}^r = \sum_{r=1}^n c_{ijr}^l c_{kst}^r, \quad 1 \leq i, j, k, s, t, l \leq n, \quad (2.11)$$

while for a weak totally coassociative 3-ary coalgebra,

$$\sum_{r=1}^n c_{rst}^l c_{ijk}^r = \sum_{r=1}^n c_{ijr}^l c_{kst}^r, \quad 1 \leq i, j, k, s, t, l \leq n. \quad (2.12)$$

Therefore, we infer the following statement.

**THEOREM 2.7.** *Let  $\mathcal{A}$  be an  $n$ -dimensional vector space with a basis  $e_1, \dots, e_n$ , and  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$  be defined as (2.9). Then,*

- (1)  $(\mathcal{A}, \Delta)$  is a partially coassociative 3-ary coalgebra if and only if the constants  $c_{ijk}^l$ ,  $1 \leq i, j, k \leq n$  satisfy the identity (2.10);
- (2)  $(\mathcal{A}, \Delta)$  is a totally coassociative 3-ary coalgebra if and only if the constants  $c_{ijk}^l$ ,  $1 \leq i, j, k \leq n$  satisfy the identity (2.11);
- (3)  $(\mathcal{A}, \Delta)$  is a weak totally coassociative 3-ary coalgebra if and only if the constants  $c_{ijk}^l$ ,  $1 \leq i, j, k \leq n$  satisfy the identity (2.12).

Now, let  $(\mathcal{A}, \mu)$  be a partially associative 3-ary algebra with a basis  $e_1, e_2, \dots, e_n$ , and the multiplication  $\mu$  of  $\mathcal{A}$  in this basis be defined as follows:

$$\mu(e_r, e_s, e_t) = \sum_{l=1}^n c_{rst}^l e_l, \quad c_{rst}^l \in \mathcal{K}, \quad 1 \leq r, s, t \leq n. \quad (2.13)$$

Using the condition (1.3), we have:

$$\begin{aligned}
& \mu(\mu(e_r, e_s, e_t), e_i, e_j) + \mu(e_r, \mu(e_s, e_t, e_i), e_j) + \mu(e_r, e_s, \mu(e_t, e_i, e_j)) \\
& = \mu\left(\sum_{l=1}^n c_{rst}^l e_l, e_i, e_j\right) + \mu\left(e_r, \sum_{l=1}^n c_{sti}^l e_l, e_j\right) + \mu\left(e_r, e_s, \sum_{l=1}^n c_{tij}^l e_l\right) \\
& = \sum_{k=1}^n \sum_{l=1}^n c_{rst}^l c_{lij}^k e_k + \sum_{k=1}^n \sum_{l=1}^n c_{sti}^l c_{rlj}^k e_k + \sum_{k=1}^n \sum_{l=1}^n c_{tij}^l c_{rst}^k e_k \\
& = 0
\end{aligned}$$

yielding  $\sum_{l=1}^n (c_{rst}^l c_{lij}^k + c_{sti}^l c_{rlj}^k + c_{tij}^l c_{rst}^k) = 0$ , i. e.  $\{c_{i_1 i_2 i_3, 1 \leq i_1, i_2, i_3 \leq n}^l\}$  satisfies the identity (2.10).

Similarly, for a totally associative 3-ary algebra and a weak totally associative 3-ary algebra, we derive, respectively:

$\sum_{l=1}^n c_{rst}^l c_{lij}^k = \sum_{l=1}^n c_{sti}^l c_{rlj}^k = \sum_{l=1}^n c_{tij}^l c_{rst}^k$ , i. e.  $\{c_{i_1 i_2 i_3, 1 \leq i_1, i_2, i_3 \leq n}^l\}$  satisfies the identity (2.11),

and

$\sum_{l=1}^n c_{rst}^l c_{lij}^k = \sum_{l=1}^n c_{tij}^l c_{rst}^k$ , i. e.  $\{c_{i_1, i_2, i_3, 1 \leq i_1, i_2, i_3 \leq n}^l\}$  satisfies the identity (2.12).

Let  $\mathcal{A}^*$  be the dual space of partially associative 3-ary algebra  $(\mathcal{A}, \mu)$ , and  $e_1^*, \dots, e_n^*$  be the dual basis of  $e_1, \dots, e_n$ ,  $\langle e_i^*, e_j \rangle = \delta_{ij}$ ,  $1 \leq i, j \leq n$ . Assume that  $\mu^* : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$  is the dual mapping of  $\mu$  defined by (2.4). Then, for every  $1 \leq l \leq n$ , we have

$$\mu^*(e_l^*) = \sum_{1 \leq r, s, t \leq n} c_{rst}^l e_r^* \otimes e_s^* \otimes e_t^*. \quad (2.14)$$

Following the identities (2.9) and (2.10),  $(\mathcal{A}^*, \mu^*)$  is a partially coassociative 3-ary coalgebra.

Conversely, if  $(\mathcal{A}, \Delta)$  is a partially coassociative 3-ary coalgebra with a basis  $e_1, \dots, e_n$  satisfying (2.9),  $\mathcal{A}^*$  is the dual space of  $\mathcal{A}$  with the dual basis  $e_1^*, \dots, e_n^*$ . Then, the dual mapping  $\Delta^* : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$  of  $\Delta$  satisfies, for every  $\xi, \eta, \gamma \in \mathcal{A}^*, x \in \mathcal{A}$ ,

$$\langle \Delta^*(\xi, \eta, \gamma), x \rangle = \langle \xi \otimes \eta \otimes \gamma, \Delta(x) \rangle. \quad (2.15)$$

Then,  $\Delta^*(e_r^*, e_s^*, e_t^*) = \sum_{l=1}^n c_{rst}^l e_l^* \in \mathcal{K}$ ,  $1 \leq r, s, t, l \leq n$  and  $\Delta^*$  satisfies identity (2.10).

REMARK 2.8. *The above constructions and discussions on the totally associative 3-ary algebra and weak totally associative 3-ary algebra also remain valid in this case.*

Therefore, the following results are true.

THEOREM 2.9. *Let  $\mathcal{A}$  be a vector space over a field  $\mathcal{K}$ , and  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ . Then,*

- (1)  *$(\mathcal{A}, \Delta)$  is a partially coassociative 3-ary coalgebra if and only if  $(\mathcal{A}^*, \Delta^*)$  is a partially associative 3-ary algebra.*
- (2)  *$(\mathcal{A}, \Delta)$  is a totally coassociative 3-ary coalgebra if and only if  $(\mathcal{A}^*, \Delta^*)$  is a totally associative 3-ary algebra.*
- (3)  *$(\mathcal{A}, \Delta)$  is a weak totally coassociative 3-ary coalgebra if and only if  $(\mathcal{A}^*, \Delta^*)$  is a weak totally associative 3-ary algebra.*

We can also give an equivalence description of (2.9) as below.

THEOREM 2.10. *Let  $\mathcal{A}$  be a vector space over a field  $\mathcal{K}$ , and  $\mu : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  be a trilinear mapping. Then,*

- (1)  *$(\mathcal{A}, \mu)$  is a partially associative 3-ary algebra if and only if  $(\mathcal{A}^*, \mu^*)$  is a partially coassociative 3-ary coalgebra.*
- (2)  *$(\mathcal{A}, \mu)$  is a totally associative 3-ary algebra if and only if  $(\mathcal{A}^*, \mu^*)$  is a totally coassociative 3-ary coalgebra.*
- (3)  *$(\mathcal{A}, \mu)$  is a weak totally associative 3-ary algebra if and only if  $(\mathcal{A}^*, \mu^*)$  is a weak totally coassociative 3-ary coalgebra.*

EXAMPLE 2.11. *Let  $(\mathcal{T}^*, \mu^*)$  be the dual of totally associative 3-ary algebra  $(\mathcal{T}, \mu)$  in Example (1.2). The product  $\mu^*$  on  $\mathcal{T}^*$  is given by*

$$\begin{aligned} \mu^*(e_1^*) &= e_1^* \otimes e_1^* \otimes e_1^* + e_1^* \otimes e_2^* \otimes e_2^* + e_2^* \otimes e_2^* \otimes e_1^* + e_2^* \otimes e_2^* \otimes e_2^* + e_2^* \otimes e_1^* \otimes e_2^* \\ \mu^*(e_2^*) &= e_1^* \otimes e_1^* \otimes e_2^* + e_1^* \otimes e_2^* \otimes e_2^* + e_2^* \otimes e_1^* \otimes e_1^* + e_2^* \otimes e_2^* \otimes e_1^* + e_2^* \otimes e_1^* \otimes e_2^* + \\ &\quad e_1^* \otimes e_2^* \otimes e_1^* + 2e_2^* \otimes e_2^* \otimes e_2^*. \end{aligned}$$

$(\mathcal{T}^*, \mu^*)$  is a totally coassociative 3-ary coalgebra.

EXAMPLE 2.12. *Let  $(\mathcal{P}^*, \mu^*)$  be the dual of partially associative 3-ary algebra  $(\mathcal{P}, \mu)$  in Example (1.6). The product  $\mu^*$  on  $\mathcal{P}^*$  is given by*

$$\mu^*(e_2^*) = e_1^* \otimes e_1^* \otimes e_1^*; \quad \mu(e_1^*) = 0.$$

$(\mathcal{P}^*, \mu^*)$  is a partially coassociative 3-ary coalgebra.

Let us recall that a coassociative 3-ary coalgebra is a partially coassociative 3-ary coalgebra, or a totally coassociative 3-ary coalgebra, or a weak totally coassociative 3-ary coalgebra.

DEFINITION 2.13. *Let  $(\mathcal{A}_1, \Delta_1)$  and  $(\mathcal{A}_2, \Delta_2)$  be two coassociative 3-ary coalgebras. If there is a linear isomorphism  $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  satisfying*

$$(\varphi \otimes \varphi \otimes \varphi)(\Delta_1(e)) = \Delta_2(\varphi(e)), \text{ for every } e \in \mathcal{A}_1, \quad (2.16)$$

then  $(\mathcal{A}_1, \Delta_1)$  is isomorphic to  $(\mathcal{A}_2, \Delta_2)$ , and  $\varphi$  is called a coassociative 3-ary coalgebra isomorphism, where

$$(\varphi \otimes \varphi \otimes \varphi) \sum_i (a_i \otimes b_i \otimes c_i) = \sum_i \varphi(a_i) \otimes \varphi(b_i) \otimes \varphi(c_i). \quad (2.17)$$

**THEOREM 2.14.** *Let  $(\mathcal{A}_1, \Delta_1)$  and  $(\mathcal{A}_2, \Delta_2)$  be two coassociative 3-ary coalgebras. Then,  $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  is a coassociative 3-ary coalgebra isomorphism from  $(\mathcal{A}_1, \Delta_1)$  to  $(\mathcal{A}_2, \Delta_2)$  if and only if the dual mapping  $\varphi^* : \mathcal{A}_2^* \rightarrow \mathcal{A}_1^*$  is an associative 3-ary algebra isomorphism from  $(\mathcal{A}_2^*, \Delta_2^*)$  to  $(\mathcal{A}_1^*, \Delta_1^*)$ , where for every  $\xi \in \mathcal{A}_2^*, v \in \mathcal{A}_1, \langle \varphi^*(\xi), v \rangle = \langle \xi, \varphi(v) \rangle$ .*

**Proof:**

Let  $(\mathcal{A}_1, \Delta_1)$  and  $(\mathcal{A}_2, \Delta_2)$  be two coassociative 3-ary coalgebras. It follows that  $(\mathcal{A}_1^*, \Delta_1^*)$  and  $(\mathcal{A}_2^*, \Delta_2^*)$  are two associative 3-ary algebras. Let  $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  be a coassociative 3-ary coalgebra isomorphism from  $(\mathcal{A}_1, \Delta_1)$  to  $(\mathcal{A}_2, \Delta_2)$ . Hence, the dual mapping  $\varphi^* : \mathcal{A}_2^* \rightarrow \mathcal{A}_1^*$  is a linear isomorphism and for every  $\xi, \eta, \gamma \in \mathcal{A}_2^*, x \in \mathcal{A}_1$  :

$$\begin{aligned} \langle \varphi^* \Delta_2^*(\xi, \eta, \gamma), x \rangle &= \langle \xi \otimes \eta \otimes \gamma, \Delta_2(\varphi(x)) \rangle \\ &= \langle \xi \otimes \eta \otimes \gamma, (\varphi \otimes \varphi \otimes \varphi) \Delta_1(x) \rangle \\ &= \langle \varphi^*(\xi) \otimes \varphi^*(\eta) \otimes \varphi^*(\gamma), \Delta_1(x) \rangle \\ &= \langle \Delta_1^*(\varphi^*(\xi), \varphi^*(\eta), \varphi^*(\gamma)), x \rangle. \end{aligned}$$

Then,  $\varphi^* \Delta_2^*(\xi, \eta, \gamma) = \Delta_1^*(\varphi^*(\xi), \varphi^*(\eta), \varphi^*(\gamma))$ , that is,  $\varphi^*$  is an associative 3-ary algebra isomorphism.  $\square$

### 3. Trimodules and matched pairs of associative 3-ary algebras

The concept of trimodule is a particular case of the concept of module over an algebra over an operad defined in [14]. For the more general context of  $n$ -ary algebras, see [15].

#### 3.1. Trimodules and matched pairs of totally associative 3-ary algebras.

**DEFINITION 3.1.** *A trimodule structure over totally associative 3-ary algebra  $(\mathcal{A}, \mu)$  on a vector space  $V$  is defined by the following three linear multiplication mappings:*

$$\begin{aligned} \mathcal{L}_\mu &: \mathcal{A} \otimes \mathcal{A} \otimes V \rightarrow V, \\ \mathcal{R}_\mu &: V \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow V, \\ \mathcal{M}_\mu &: \mathcal{A} \otimes V \otimes \mathcal{A} \rightarrow V \end{aligned}$$

satisfying the following compatibility conditions

$$\mathcal{L}_\mu(a, b)(\mathcal{L}_\mu(c, d)(v)) = \mathcal{L}_\mu(\mu(a, b, c), d)(v) = \mathcal{L}_\mu(a, \mu(b, c, d))(v), \quad (3.1)$$

$$\mathcal{R}_\mu(c, d)(\mathcal{R}_\mu(a, b)(v)) = \mathcal{R}_\mu(a, \mu(b, c, d))(v) = \mathcal{R}_\mu(\mu(a, b, c), d)(v), \quad (3.2)$$

$$\mathcal{M}_\mu(a, z)(\mathcal{M}_\mu(b, y)(\mathcal{M}_\mu(c, x)(v))) = \mathcal{M}_\mu(\mu(a, b, c), \mu(x, y, z))(v), \quad (3.3)$$

$$\mathcal{M}_\mu(a, d)(\mathcal{L}_\mu(b, c)(v)) = \mathcal{L}_\mu(a, b)(\mathcal{M}_\mu(c, d)(v)) = \mathcal{M}_\mu(\mu(a, b, c), d)(v), \quad (3.4)$$

$$\mathcal{M}_\mu(a, d)(\mathcal{R}_\mu(b, c)(v)) = \mathcal{R}_\mu(c, d)(\mathcal{M}_\mu(a, b)(v)) = \mathcal{M}_\mu(a, \mu(b, c, d))(v), \quad (3.5)$$

$$\mathcal{R}_\mu(c, d)(\mathcal{L}_\mu(a, b)(v)) = \mathcal{L}_\mu(a, b)(\mathcal{R}_\mu(c, d)(v)) = \mathcal{M}_\mu(a, d)(\mathcal{M}_\mu(b, c)(v)), \quad (3.6)$$

$\forall a, b, c, d, x, y, z \in \mathcal{A}, v \in V$ .

**PROPOSITION 3.2.**  *$(\mathcal{L}_\mu, \mathcal{M}_\mu, \mathcal{R}_\mu, V)$  is a trimodule of an associative totally 3-ary algebra  $(\mathcal{A}, \mu)$  if and only if the direct sum  $(\mathcal{A} \oplus V, \tau)$  of the underlying vector spaces of  $\mathcal{A}$  and  $V$  is turned into an associative totally 3-ary algebra  $\tau$  given by*

$$\tau[(x + a), (y + b), (z + c)] = \mu(x, y, z) + \mathcal{L}_\mu(x, y)(c) + \mathcal{M}_\mu(x, z)(b) + \mathcal{R}_\mu(y, z)(a),$$

for all  $x, y, z \in \mathcal{A}, a, b, c \in V$ . We denote it by  $\mathcal{A} \ltimes_{\mathcal{L}_\mu, \mathcal{M}_\mu, \mathcal{R}_\mu} V$ .

**Proof** Let  $v_1, v_2, v_3, v_4, v_5 \in V$  and  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$ . Set

$$\begin{aligned} & \tau[\tau[(x_1 + v_1), (x_2 + v_2), (x_3 + v_3)], (x_4 + v_4), (x_5 + v_5)] \\ &= \tau[(x_1 + v_1), \tau[(x_2 + v_2), (x_3 + v_3), (x_4 + v_4)], (x_5 + v_5)] \\ &= \tau[(x_1 + v_1), (x_2 + v_2), \tau[(x_3 + v_3), (x_4 + v_4), (x_5 + v_5)]] \end{aligned}$$

After computation, we obtain the Eqs.(3.1)-(3.6) while the Eq.(3.3) is satisfied with the specification of the action of  $\mathcal{M}_\mu$ . Then  $(\mathcal{L}_\mu, \mathcal{M}_\mu, \mathcal{R}_\mu, V)$  is a trimodule of the associative totally 3-ary algebra  $(\mathcal{A}, \mu)$  if and only if the direct sum  $(\mathcal{A} \oplus V, \tau)$  is a totally 3-ary algebra.  $\square$

**REMARK 3.3.** *In the case where the Eq.(3.3) is not satisfied, we refer to the name quasi trimodule structure instead of simply trimodule structure.*

**THEOREM 3.4.** *Let  $(\mathcal{A}, \mu_\mathcal{A})$  and  $(\mathcal{B}, \mu_\mathcal{B})$  be two associative totally 3-ary algebras. Suppose that there are linear maps  $\mathcal{L}_{\mu_\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}, \mathcal{R}_{\mu_\mathcal{A}} : \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{B}, \mathcal{M}_{\mu_\mathcal{A}} : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{B}$  and  $\mathcal{L}_{\mu_\mathcal{B}} : \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A}, \mathcal{R}_{\mu_\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{A}, \mathcal{M}_{\mu_\mathcal{B}} : \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$  such that  $(\mathcal{L}_{\mu_\mathcal{A}}, \mathcal{M}_{\mu_\mathcal{A}}, \mathcal{R}_{\mu_\mathcal{A}}, \mathcal{B})$  is a quasi trimodule of the associative totally 3-ary algebra  $(\mathcal{A}, \mu_\mathcal{A})$  and  $(\mathcal{L}_{\mu_\mathcal{B}}, \mathcal{M}_{\mu_\mathcal{B}}, \mathcal{R}_{\mu_\mathcal{B}}, \mathcal{A})$  is a quasi trimodule of the associative totally 3-ary algebra  $(\mathcal{B}, \mu_\mathcal{B})$  and they satisfy the following conditions:*

$$\mu_\mathcal{A}(\mathcal{L}_{\mu_\mathcal{B}}(a, b)(x), y, z) = \mathcal{L}_{\mu_\mathcal{B}}[a, \mathcal{R}_{\mu_\mathcal{A}}(x, y)(b)](z) = \mathcal{L}_{\mu_\mathcal{B}}(a, b)(\mu_\mathcal{A}(x, y, z)), \quad (3.7)$$

$$\mu_\mathcal{A}(\mathcal{M}_{\mu_\mathcal{B}}(a, b)(x), y, z) = \mathcal{L}_{\mu_\mathcal{B}}[a, \mathcal{M}_{\mu_\mathcal{A}}(x, y)(b)](z) = \mathcal{M}_{\mu_\mathcal{B}}[a, \mathcal{R}_{\mu_\mathcal{A}}(y, z)(b)](x), \quad (3.8)$$

$$\mu_\mathcal{A}(\mathcal{R}_{\mu_\mathcal{B}}(a, b)(x), y, z) = \mu_\mathcal{A}[x, \mathcal{L}_{\mu_\mathcal{B}}(a, b)(y), z] = \mathcal{R}_{\mu_\mathcal{B}}[a, \mathcal{R}_{\mu_\mathcal{A}}(y, z)(b)](x), \quad (3.9)$$

$$\mathcal{L}_{\mu_\mathcal{B}}[\mathcal{L}_{\mu_\mathcal{A}}(x, y)(a), b](z) = \mu_\mathcal{A}[x, \mathcal{R}_{\mu_\mathcal{B}}(a, b)(y), z] = \mu_\mathcal{A}[x, y, \mathcal{L}_{\mu_\mathcal{B}}(a, b)(z)], \quad (3.10)$$

$$\mathcal{L}_{\mu_\mathcal{B}}[\mathcal{M}_{\mu_\mathcal{A}}(x, y)(a), b](z) = \mu_\mathcal{A}[x, \mathcal{M}_{\mu_\mathcal{B}}(a, b)(y), z] = \mathcal{R}_{\mu_\mathcal{B}}[a, \mathcal{M}_{\mu_\mathcal{A}}(y, z)(b)](x), \quad (3.11)$$

$$\mathcal{L}_{\mu_\mathcal{B}}[\mathcal{R}_{\mu_\mathcal{A}}(x, y)(a), b](z) = \mathcal{L}_{\mu_\mathcal{B}}[a, \mathcal{L}_{\mu_\mathcal{A}}(x, y)(b)](z) = \mathcal{M}_{\mu_\mathcal{B}}[a, \mathcal{M}_{\mu_\mathcal{A}}(y, z)(b)](x), \quad (3.12)$$

$$\mathcal{M}_{\mu_\mathcal{B}}[\mathcal{L}_{\mu_\mathcal{A}}(x, y)(a), b](z) = \mathcal{R}_{\mu_\mathcal{B}}[\mathcal{M}_{\mu_\mathcal{A}}(y, z)(a), b](x) = \mu_\mathcal{A}[x, y, \mathcal{M}_{\mu_\mathcal{B}}(a, b)(z)], \quad (3.13)$$

$$\mathcal{M}_{\mu_\mathcal{B}}[\mathcal{M}_{\mu_\mathcal{A}}(x, y)(a), b](z) = \mathcal{R}_{\mu_\mathcal{B}}[\mathcal{R}_{\mu_\mathcal{A}}(y, z)(a), b](x) = \mathcal{R}_{\mu_\mathcal{B}}[a, \mathcal{L}_{\mu_\mathcal{A}}(y, z)(b)](x), \quad (3.14)$$

$$\mathcal{M}_{\mu_\mathcal{B}}[\mathcal{R}_{\mu_\mathcal{A}}(x, y)(a), b](z) = \mathcal{M}_{\mu_\mathcal{B}}(a, b)(\mu_\mathcal{A}(x, y, z)) = \mathcal{M}_{\mu_\mathcal{B}}[a, \mathcal{L}_{\mu_\mathcal{A}}(y, z)(b)](x), \quad (3.15)$$

$$\mathcal{R}_{\mu_\mathcal{B}}(a, b)(\mu_\mathcal{A}(x, y, z)) = \mathcal{R}_{\mu_\mathcal{B}}[\mathcal{L}_{\mu_\mathcal{A}}(y, z)(a), b](x) = \mu_\mathcal{A}[x, y, \mathcal{R}_{\mu_\mathcal{B}}(a, b)(z)], \quad (3.16)$$

$$\mu_\mathcal{B}[\mathcal{L}_{\mu_\mathcal{A}}(x, y)(a), b, c] = \mathcal{L}_{\mu_\mathcal{A}}[x, \mathcal{R}_{\mu_\mathcal{B}}(a, b)(y)](c) = \mathcal{L}_{\mu_\mathcal{A}}(x, y)(\mu_\mathcal{B}(a, b, c)), \quad (3.17)$$

$$\mu_\mathcal{B}[\mathcal{M}_{\mu_\mathcal{A}}(x, y)(a), b, c] = \mathcal{L}_{\mu_\mathcal{A}}[x, \mathcal{M}_{\mu_\mathcal{B}}(a, b)(y)](c) = \mathcal{M}_{\mu_\mathcal{A}}[x, \mathcal{R}_{\mu_\mathcal{B}}(b, c)(y)](a), \quad (3.18)$$

$$\mu_\mathcal{B}[\mathcal{R}_{\mu_\mathcal{A}}(x, y)(a), b, c] = \mu_\mathcal{B}[a, \mathcal{L}_{\mu_\mathcal{A}}(x, y)(b), c] = \mathcal{R}_{\mu_\mathcal{A}}[x, \mathcal{R}_{\mu_\mathcal{B}}(b, c)(y)](a), \quad (3.19)$$

$$\mathcal{L}_{\mu_\mathcal{A}}[\mathcal{L}_{\mu_\mathcal{B}}(a, b)(x), y](c) = \mu_\mathcal{B}[a, \mathcal{R}_{\mu_\mathcal{A}}(x, y)(b), c] = \mu_\mathcal{B}[a, b, \mathcal{L}_{\mu_\mathcal{A}}(x, y)(c)], \quad (3.20)$$

$$\mathcal{L}_{\mu_\mathcal{A}}[\mathcal{M}_{\mu_\mathcal{B}}(a, b)(x), y](c) = \mu_\mathcal{B}[a, \mathcal{M}_{\mu_\mathcal{A}}(x, y)(b), c] = \mathcal{R}_{\mu_\mathcal{A}}[x, \mathcal{M}_{\mu_\mathcal{B}}(b, c)(y)](a), \quad (3.21)$$

$$\mathcal{L}_{\mu_\mathcal{A}}[\mathcal{R}_{\mu_\mathcal{B}}(a, b)(x), y](c) = \mathcal{L}_{\mu_\mathcal{A}}[x, \mathcal{L}_{\mu_\mathcal{B}}(a, b)(y)](c) = \mathcal{M}_{\mu_\mathcal{A}}[x, \mathcal{M}_{\mu_\mathcal{B}}(b, c)(y)](a), \quad (3.22)$$

$$\mathcal{M}_{\mu_\mathcal{A}}[\mathcal{L}_{\mu_\mathcal{B}}(a, b)(x), y](c) = \mathcal{R}_{\mu_\mathcal{A}}[\mathcal{M}_{\mu_\mathcal{B}}(b, c)(x), y](a) = \mu_\mathcal{B}[a, b, \mathcal{M}_{\mu_\mathcal{A}}(x, y)(c)], \quad (3.23)$$

$$\mathcal{M}_{\mu_\mathcal{A}}[\mathcal{M}_{\mu_\mathcal{B}}(a, b)(x), y](c) = \mathcal{R}_{\mu_\mathcal{A}}[\mathcal{R}_{\mu_\mathcal{B}}(b, c)(x), y](a) = \mathcal{R}_{\mu_\mathcal{A}}[x, \mathcal{L}_{\mu_\mathcal{B}}(b, c)(y)](a), \quad (3.24)$$

$$\mathcal{M}_{\mu_\mathcal{A}}[\mathcal{R}_{\mu_\mathcal{B}}(a, b)(x), y](c) = \mathcal{M}_{\mu_\mathcal{A}}(x, y)(\mu_\mathcal{B}(a, b, c)) = \mathcal{M}_{\mu_\mathcal{A}}[x, \mathcal{L}_{\mu_\mathcal{B}}(b, c)(y)](a), \quad (3.25)$$

$$\mathcal{R}_{\mu_\mathcal{A}}(x, y)(\mu_\mathcal{B}(a, b, c)) = \mathcal{R}_{\mu_\mathcal{A}}[\mathcal{L}_{\mu_\mathcal{B}}(b, c)(x), y](a) = \mu_\mathcal{B}[a, b, \mathcal{R}_{\mu_\mathcal{A}}(x, y)(c)], \quad (3.26)$$



for any  $x, y, z \in \mathcal{A}, a, b, c \in \mathcal{B}$ . Then, there is an associative totally 3-ary algebra structure on the direct sum  $\mathcal{A} \oplus \mathcal{B}$  of the underlying vector spaces of  $\mathcal{A}$  and  $\mathcal{B}$  given by the product  $\tau$  defined by

$$\begin{aligned} \tau[(x+a), (y+b), (z+c)] &= [\mu_{\mathcal{A}}(x, y, z) + \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(z) + \mathcal{M}_{\mu_{\mathcal{B}}}(a, c)(y) + \mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(x)] + \\ &[\mu_{\mathcal{B}}(a, b, c) + \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(c) + \mathcal{M}_{\mu_{\mathcal{A}}}(x, z)(b) + \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(a)] \end{aligned}$$

for any  $x, y, z \in \mathcal{A}, a, b, c \in \mathcal{B}$ . Let  $\mathcal{A} \bowtie_{\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}}^{\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}} \mathcal{B}$  denote this associative totally 3-ary algebra.

**Proof** Let  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$  and  $y_1, y_2, y_3, y_4, y_5 \in \mathcal{B}$ . By definition, we have

$$\begin{aligned} \tau[(x+a), (y+b), (z+c)] &= [\mu_{\mathcal{A}}(x, y, z) + \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(z) + \mathcal{M}_{\mu_{\mathcal{B}}}(a, c)(y) + \mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(x)] + \\ &[\mu_{\mathcal{B}}(a, b, c) + \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(c) + \mathcal{M}_{\mu_{\mathcal{A}}}(x, z)(b) + \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(a)] \end{aligned}$$

for any  $x, y, z \in \mathcal{A}, a, b, c \in \mathcal{B}$ . Setting the strong condition

$$\begin{aligned} &\tau[\tau[(x_1+y_1), (x_2+y_2), (x_3+y_3)], (x_4+y_4), (x_5+y_5)] \\ &= \tau[(x_1+y_1), \tau[(x_2+y_2), (x_3+y_3), (x_4+y_4)], (x_5+y_5)] \\ &= \tau[(x_1+y_1), (x_2+y_2), \tau[(x_3+y_3), (x_4+y_4), (x_5+y_5)]], \end{aligned}$$

we obtain by direct computation the Eqs.(3.7) - (3.26). Then, there is an associative totally 3-ary algebra structure on the direct sum  $\mathcal{A} \oplus \mathcal{B}$  of the underlying vector spaces of  $\mathcal{A}$  and  $\mathcal{B}$  if and only if the Eqs.(3.7) - (3.26) are satisfied.  $\square$

**DEFINITION 3.5.** Let  $(\mathcal{A}, \mu_{\mathcal{A}})$  and  $(\mathcal{B}, \mu_{\mathcal{B}})$  be two totally associative 3-ary algebras. Suppose that there are linear maps  $\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}$  and  $\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}$  such that  $(\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{B})$  is a quasi trimodule of  $\mathcal{A}$  and  $(\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}, \mathcal{A})$  is a quasi trimodule of  $\mathcal{B}$ . If Eqs.3.7 - 3.26 are satisfied, then  $(\mathcal{A}, \mathcal{B}, \mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}})$  is called a matched pair of totally associative 3-ary algebras.

**DEFINITION 3.6.** Let  $(\mathcal{A}, \mu_{\mathcal{A}})$  and  $(\mathcal{B}, \mu_{\mathcal{B}})$  be two totally associative 3-ary algebras. Suppose that there are linear maps  $\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}$  and  $\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}$  such that  $(\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{B})$  is a trimodule of  $\mathcal{A}$  and  $(\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}, \mathcal{A})$  is a trimodule of  $\mathcal{B}$ .

Then  $(\mathcal{A}, \mathcal{B}, \mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}})$  is a matched pair of totally associative 3-ary algebras if Eqs.3.7 - 3.26 are satisfied and if the following conditions also hold:

$$\mathcal{M}_{\mu_{\mathcal{A}}}(a, z)(\mathcal{M}_{\mu_{\mathcal{A}}}(b, y)(\mathcal{M}_{\mu_{\mathcal{A}}}(c, x)(v))) = \mathcal{M}_{\mu_{\mathcal{A}}}(\mu_{\mathcal{A}}(a, b, c), \mu_{\mathcal{A}}(x, y, z))(v), \quad (3.27)$$

$$\mathcal{M}_{\mu_{\mathcal{B}}}(a', z')(\mathcal{M}_{\mu_{\mathcal{B}}}(b', y')(\mathcal{M}_{\mu_{\mathcal{B}}}(c', x')(v))) = \mathcal{M}_{\mu_{\mathcal{B}}}(\mu_{\mathcal{B}}(a', b', c'), \mu_{\mathcal{B}}(x', y', z'))(v), \quad (3.28)$$

for any  $x, y, z, a, b, c, v \in \mathcal{A}, x', y', z', a', b', c', v' \in \mathcal{B}$ .

**LEMMA 3.7.** Let  $(\mathcal{L}_{\mu}, \mathcal{M}_{\mu}, \mathcal{R}_{\mu})$  be a trimodule of a totally associative 3-ary algebra  $\mathcal{A}$ . Then, the linear maps  $\mathcal{L}_{\mu}^*, \mathcal{M}_{\mu}^*, \mathcal{R}_{\mu}^* : \mathcal{A} \otimes \mathcal{A} \rightarrow gl(V^*)$  given by

$$\begin{aligned} \langle \mathcal{L}_{\mu}^*(x, y)u^*, v \rangle &= \langle \mathcal{L}_{\mu}(x, y)v, u^* \rangle; \langle \mathcal{M}_{\mu}^*(x, y)u^*, v \rangle = \langle \mathcal{M}_{\mu}(x, y)v, u^* \rangle; \\ \langle \mathcal{R}_{\mu}^*(x, y)u^*, v \rangle &= \langle \mathcal{R}_{\mu}(x, y)v, u^* \rangle; \text{ for all } x, y \in \mathcal{A}, v \in V, u^* \in V^* \end{aligned}$$

realize a trimodule of  $\mathcal{A}$  denoted by  $(\mathcal{R}_{\mu}^*, \mathcal{M}_{\mu}^*, \mathcal{L}_{\mu}^*)$  satisfying

$$\mathcal{R}_{\mu}^*(a, b)(\mathcal{R}_{\mu}^*(c, d)(u^*)) = \mathcal{R}_{\mu}^*(\mu(a, b, c), d)(u^*) = \mathcal{R}_{\mu}^*(a, \mu(b, c, d))(u^*), \quad (3.29)$$

$$\mathcal{L}_{\mu}^*(c, d)(\mathcal{L}_{\mu}^*(a, b)(u^*)) = \mathcal{L}_{\mu}^*(a, \mu(b, c, d))(u^*) = \mathcal{L}_{\mu}^*(\mu(a, b, c), d)(u^*), \quad (3.30)$$

$$\mathcal{M}_{\mu}^*(a, z)(\mathcal{M}_{\mu}^*(b, y)(\mathcal{M}_{\mu}^*(c, x)(u^*))) = \mathcal{M}_{\mu}^*(\mu(c, b, a), \mu(z, y, x))(u^*), \quad (3.31)$$

$$\mathcal{M}_{\mu}^*(a, d)(\mathcal{R}_{\mu}^*(b, c)(u^*)) = \mathcal{R}_{\mu}^*(d, b)(\mathcal{M}_{\mu}^*(a, c)(u^*)) = \mathcal{M}_{\mu}^*(a, \mu(d, b, c))(u^*), \quad (3.32)$$

$$\mathcal{M}_{\mu}^*(a, d)(\mathcal{R}_{\mu}^*(b, c)(u^*)) = \mathcal{L}_{\mu}^*(c, a)(\mathcal{M}_{\mu}^*(b, d)(u^*)) = \mathcal{M}_{\mu}^*(\mu(b, c, a), d)(u^*), \quad (3.33)$$

$$\mathcal{L}_{\mu}^*(c, d)(\mathcal{R}_{\mu}^*(a, b)(u^*)) = \mathcal{R}_{\mu}^*(a, b)(\mathcal{L}_{\mu}^*(c, d)(u^*)) = \mathcal{M}_{\mu}^*(d, a)(\mathcal{M}_{\mu}^*(c, b)(u^*)), \quad (3.34)$$

$\forall a, b, c, d, x, y, z \in \mathcal{A}, u^* \in V^*$ .

**Proof:** By a direct computation, we obtain the result.  $\square$

Let us now give some notations useful in the sequel. Let  $(\mathcal{A}, \mu)$  be an associative 3-ary algebra.

Considering the linear maps of the left  $L_\mu$ , right  $R_\mu$  and central  $M_\mu$  multiplication operations defined as:

$$\begin{aligned} L_\mu : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ (x, y) &\longmapsto L_\mu(x, y) : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ z & \longmapsto & \mu(x, y, z), \end{array} \end{aligned} \quad (3.35)$$

$$\begin{aligned} R_\mu : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ (x, y) &\longmapsto R_\mu(x, y) : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ z & \longmapsto & \mu(z, x, y), \end{array} \end{aligned} \quad (3.36)$$

$$\begin{aligned} M_\mu : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ (x, y) &\longmapsto M_\mu(x, y) : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ z & \longmapsto & \mu(x, z, y). \end{array} \end{aligned} \quad (3.37)$$

The dual maps  $L_\mu^*, R_\mu^*, M_\mu^*$  of the linear maps  $L_\mu, R_\mu, M_\mu$  are defined, respectively, as:  $L_\mu^*, R_\mu^*, M_\mu^* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A}^*)$  such that:

$$\begin{aligned} L_\mu^* : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}^*) \\ (x, y) &\longmapsto L_\mu^*(x, y) : \begin{array}{ccc} \mathcal{A}^* & \longrightarrow & \mathcal{A}^* \\ u^* & \longmapsto & L_\mu^*(x, y)(u^*) : \end{array} \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{K} \\ v & \longmapsto & \langle L_\mu^*(x, y)(u^*), v \rangle \\ & & := \langle u^*, L_\mu(x, y)v \rangle, \end{array} \end{aligned} \quad (3.38)$$

$$\begin{aligned} R_\mu^* : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}^*) \\ (x, y) &\longmapsto R_\mu^*(x, y) : \begin{array}{ccc} \mathcal{A}^* & \longrightarrow & \mathcal{A}^* \\ u^* & \longmapsto & R_\mu^*(x, y)(u^*) : \end{array} \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{K} \\ v & \longmapsto & \langle R_\mu^*(x, y)(u^*), v \rangle \\ & & := \langle u^*, R_\mu(x, y)v \rangle, \end{array} \end{aligned} \quad (3.39)$$

$$\begin{aligned} M_\mu^* : \mathcal{A} \otimes \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}^*) \\ (x, y) &\longmapsto M_\mu^*(x, y) : \begin{array}{ccc} \mathcal{A}^* & \longrightarrow & \mathcal{A}^* \\ u^* & \longmapsto & M_\mu^*(x, y)(u^*) : \end{array} \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{K} \\ v & \longmapsto & \langle M_\mu^*(x, y)(u^*), v \rangle \\ & & := \langle u^*, M_\mu(x, y)v \rangle, \end{array} \end{aligned} \quad (3.40)$$

for all  $x, y, z, v \in \mathcal{A}, u^* \in \mathcal{A}^*$ , where  $\mathcal{A}^*$  is the dual space of  $\mathcal{A}$ .

**PROPOSITION 3.8.** *Let  $(\mathcal{A}, \mu)$  be a totally associative 3-ary algebra. Then  $(L_\mu, 0, 0, \mathcal{A})$ ,  $(0, 0, R_\mu, \mathcal{A})$ ,  $(L_\mu, M_\mu, R_\mu, \mathcal{A})$ ,  $(R_\mu^*, 0, 0, \mathcal{A}^*)$ ,  $(0, 0, L_\mu^*, \mathcal{A}^*)$  and  $(R_\mu^*, M_\mu^*, L_\mu^*, \mathcal{A}^*)$  are trimodules of  $\mathcal{A}$ .*

**Proof:** It is straightforward.  $\square$

For a linear map  $\phi : V_1 \rightarrow V_2$ , we denote the dual (linear) map by  $\phi^* : V_2^* \rightarrow V_1^*$  given by

$$\langle v, \phi^*(u^*) \rangle = \langle \phi(v), u^* \rangle \text{ for all } v \in V_1, u^* \in V_2^*.$$

**THEOREM 3.9.** *Let  $(\mathcal{A}, \mu)$  be a totally associative 3-ary algebra. Suppose that there is a totally associative 3-ary algebra structure  $\nu$  on its dual space  $\mathcal{A}^*$ . Then,  $(\mathcal{A}, \mathcal{A}^*, R_\mu^*, M_\mu^*, L_\mu^*, R_\nu^*, M_\nu^*, L_\nu^*)$  is a matched pair of totally associative 3-ary algebras if and only if there is an associative totally 3-ary algebra structure on the direct sum  $\mathcal{A} \oplus \mathcal{A}^*$  of the underlying vector spaces of  $\mathcal{A}$  and  $\mathcal{A}^*$  given by the product  $\tau$  defined by*

$$\tau[(x + a^*), (y + b^*), (z + c^*)] = [\mu(x, y, z) + L_\nu^*(a^*, b^*)(z) + M_\nu^*(a^*, c^*)(y) + R_\nu^*(b^*, c^*)(x)] + [\nu(a^*, b^*, c^*) + L_\mu^*(x, y)(c^*) + M_\mu^*(x, z)(b^*) + R_\mu^*(y, z)(a^*)]$$

for any  $x, y, z \in \mathcal{A}, a^*, b^*, c^* \in \mathcal{A}^*$ .

**Proof** We hold a same reasoning as in the proof of Theorem(3.4) with  $\mathcal{B} = \mathcal{A}^*$  and

$$\begin{aligned} \tau[(x + a^*), (y + b^*), (z + c^*)] &= [\mu(x, y, z) + L_\nu^*(a^*, b^*)(z) + M_\nu^*(a^*, c^*)(y) + R_\nu^*(b^*, c^*)(x)] + \\ &\quad [\nu(a^*, b^*, c^*) + L_\mu^*(x, y)(c^*) + M_\mu^*(x, z)(b^*) + R_\mu^*(y, z)(a^*)] \end{aligned}$$

for any  $x, y, z \in \mathcal{A}, a^*, b^*, c^* \in \mathcal{A}^*$ .  $\square$

**3.2. Trimodules and matched pairs of partially associative 3-ary algebras.** In this sequel, we give a definition of trimodule for the partially associative 3-ary algebras. Then, the matched pairs of partially associative 3-ary algebras are established.

**DEFINITION 3.10.** A trimodule structure over partially associative 3-ary algebra  $(\mathcal{A}, \mu)$  on a vector space  $V$  is defined by the following three linear multiplication mappings:

$$\begin{aligned} \mathcal{L}_\mu &: \mathcal{A} \otimes \mathcal{A} \otimes V \rightarrow V, \\ \mathcal{R}_\mu &: V \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow V, \\ \mathcal{M}_\mu &: \mathcal{A} \otimes V \otimes \mathcal{A} \rightarrow V \end{aligned}$$

satisfying the following compatibility conditions

$$\mathcal{L}_\mu(a, b)(\mathcal{L}_\mu(c, d)(v)) + \mathcal{L}_\mu(\mu(a, b, c), d)(v) + \mathcal{L}_\mu(a, \mu(b, c, d))(v) = 0, \quad (3.41)$$

$$\mathcal{R}_\mu(c, d)(\mathcal{R}_\mu(a, b)(v)) + \mathcal{R}_\mu(a, \mu(b, c, d))(v) + \mathcal{R}_\mu(\mu(a, b, c), d)(v) = 0, \quad (3.42)$$

$$\mathcal{M}_\mu(a, z)(\mathcal{M}_\mu(b, y)(\mathcal{M}_\mu(c, x)(v))) = \mathcal{M}_\mu(\mu(a, b, c), \mu(x, y, z))(v), \quad (3.43)$$

$$\mathcal{M}_\mu(a, d)(\mathcal{L}_\mu(b, c)(v)) + \mathcal{L}_\mu(a, b)(\mathcal{M}_\mu(c, d)(v)) + \mathcal{M}_\mu(\mu(a, b, c), d)(v) = 0, \quad (3.44)$$

$$\mathcal{M}_\mu(a, d)(\mathcal{R}_\mu(b, c)(v)) + \mathcal{R}_\mu(c, d)(\mathcal{M}_\mu(a, b)(v)) + \mathcal{M}_\mu(a, \mu(b, c, d))(v) = 0, \quad (3.45)$$

$$\mathcal{R}_\mu(c, d)(\mathcal{L}_\mu(a, b)(v)) + \mathcal{L}_\mu(a, b)(\mathcal{R}_\mu(c, d)(v)) + \mathcal{M}_\mu(a, d)(\mathcal{M}_\mu(b, c)(v)) = 0, \quad (3.46)$$

$\forall a, b, c, d, x, y, z \in \mathcal{A}, v \in V$ .

**PROPOSITION 3.11.**  $(\mathcal{L}_\mu, \mathcal{M}_\mu, \mathcal{R}_\mu, V)$  is a trimodule of an associative partially 3-ary algebra  $(\mathcal{A}, \mu)$  if and only if the direct sum  $(\mathcal{A} \oplus V, \tau)$  of the underlying vector spaces of  $\mathcal{A}$  and  $V$  is turned into an associative partially 3-ary algebra  $\tau$  given by

$$\tau[(x + a), (y + b), (z + c)] = \mu(x, y, z) + \mathcal{L}_\mu(x, y)(c) + \mathcal{M}_\mu(x, z)(b) + \mathcal{R}_\mu(y, z)(a),$$

for all  $x, y, z \in \mathcal{A}, a, b, c \in V$ . We denote it by  $\mathcal{A} \ltimes_{\mathcal{L}_\mu, \mathcal{M}_\mu, \mathcal{R}_\mu} V$ .

**Proof** Let  $v_1, v_2, v_3, v_4, v_5 \in V$  and  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$ . Set

$$\begin{aligned} &\tau[\tau[(x_1 + v_1), (x_2 + v_2), (x_3 + v_3)], (x_4 + v_4), (x_5 + v_5)] + \\ &\tau[(x_1 + v_1), \tau[(x_2 + v_2), (x_3 + v_3), (x_4 + v_4)], (x_5 + v_5)] + \\ &\tau[(x_1 + v_1), (x_2 + v_2), \tau[(x_3 + v_3), (x_4 + v_4), (x_5 + v_5)]] = 0. \end{aligned}$$

After computation, we obtain the Eqs.(3.41)-(3.46) while the Eq.(3.43) is satisfied with the specification of the action of  $\mathcal{M}_\mu$ . Then  $(\mathcal{L}_\mu, \mathcal{M}_\mu, \mathcal{R}_\mu, V)$  is a trimodule of the associative partially 3-ary algebra  $(\mathcal{A}, \mu)$  if and only if the direct sum  $(\mathcal{A} \oplus V, \tau)$  is a partially 3-ary algebra.  $\square$

**REMARK 3.12.** In the case where the Eq.(3.43) is not satisfied, we refer to the name of quasi trimodule structure instead of simply trimodule structure.

**THEOREM 3.13.** Let  $(\mathcal{A}, \mu_{\mathcal{A}})$  and  $(\mathcal{B}, \mu_{\mathcal{B}})$  be two associative partially 3-ary algebras. Suppose that there are linear maps  $\mathcal{L}_{\mu_{\mathcal{A}}} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}, \mathcal{R}_{\mu_{\mathcal{A}}} : \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{B}, \mathcal{M}_{\mu_{\mathcal{A}}} : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{B}$  and  $\mathcal{L}_{\mu_{\mathcal{B}}} : \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A}, \mathcal{R}_{\mu_{\mathcal{B}}} : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{A}, \mathcal{M}_{\mu_{\mathcal{B}}} : \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$  such that  $(\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{B})$  is a quasi trimodule of the associative partially 3-ary algebra  $(\mathcal{A}, \mu_{\mathcal{A}})$ , and  $(\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}, \mathcal{A})$  is a quasi trimodule of the associative partially 3-ary algebra  $(\mathcal{B}, \mu_{\mathcal{B}})$ , satisfying the following conditions:

$$\mu_{\mathcal{A}}(\mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(x), y, z) + \mathcal{L}_{\mu_{\mathcal{B}}}[a, \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(b)](z) + \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(\mu_{\mathcal{A}}(x, y, z)) = 0, \quad (3.47)$$

$$\mu_{\mathcal{A}}(\mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(x), y, z) + \mathcal{L}_{\mu_{\mathcal{B}}}[a, \mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(b)](z) + \mathcal{M}_{\mu_{\mathcal{B}}}[a, \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(b)](x) = 0, \quad (3.48)$$

$$\mu_{\mathcal{A}}(\mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(x), y, z) + \mu_{\mathcal{A}}[x, \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(y), z] + \mathcal{R}_{\mu_{\mathcal{B}}}[a, \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(b)](x) = 0, \quad (3.49)$$

$$\mathcal{L}_{\mu_{\mathcal{B}}}[\mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(a), b](z) + \mu_{\mathcal{A}}[x, \mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(y), z] + \mu_{\mathcal{A}}[x, y, \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(z)] = 0, \quad (3.50)$$

$$\mathcal{L}_{\mu_{\mathcal{B}}}[\mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(a), b](z) + \mu_{\mathcal{A}}[x, \mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(y), z] + \mathcal{R}_{\mu_{\mathcal{B}}}[a, \mathcal{M}_{\mu_{\mathcal{A}}}(y, z)(b)](x) = 0, \quad (3.51)$$

$$\mathcal{L}_{\mu_{\mathcal{B}}}[\mathcal{R}_{\mu_{\mathcal{A}}}(x, y)(a), b](z) + \mathcal{L}_{\mu_{\mathcal{B}}}[a, \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(b)](z) + \mathcal{M}_{\mu_{\mathcal{B}}}[a, \mathcal{M}_{\mu_{\mathcal{A}}}(y, z)(b)](x) = 0, \quad (3.52)$$

$$\mathcal{M}_{\mu_{\mathcal{B}}}[\mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(a), b](z) + \mathcal{R}_{\mu_{\mathcal{B}}}[\mathcal{M}_{\mu_{\mathcal{A}}}(y, z)(a), b](x) + \mu_{\mathcal{A}}[x, y, \mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(z)] = 0, \quad (3.53)$$

$$\mathcal{M}_{\mu_{\mathcal{B}}}[\mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(a), b](z) + \mathcal{R}_{\mu_{\mathcal{B}}}[\mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(a), b](x) + \mathcal{R}_{\mu_{\mathcal{B}}}[a, \mathcal{L}_{\mu_{\mathcal{A}}}(y, z)(b)](x) = 0, \quad (3.54)$$

$$\mathcal{M}_{\mu_{\mathcal{B}}}[\mathcal{R}_{\mu_{\mathcal{A}}}(x, y)(a), b](z) + \mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(\mu_{\mathcal{A}}(x, y, z) + \mathcal{M}_{\mu_{\mathcal{B}}}[a, \mathcal{L}_{\mu_{\mathcal{A}}}(y, z)(b)](x) = 0, \quad (3.55)$$

$$\mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(\mu_{\mathcal{A}}(x, y, z)) + \mathcal{R}_{\mu_{\mathcal{B}}}[\mathcal{L}_{\mu_{\mathcal{A}}}(y, z)(a), b](x) + \mu_{\mathcal{A}}[x, y, \mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(z)] = 0, \quad (3.56)$$

$$\mu_{\mathcal{B}}[\mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(a), b, c] + \mathcal{L}_{\mu_{\mathcal{A}}}[x, \mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(y)](c) + \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(\mu_{\mathcal{B}}(a, b, c)) = 0, \quad (3.57)$$

$$\mu_{\mathcal{B}}[\mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(a), b, c] + \mathcal{L}_{\mu_{\mathcal{A}}}[x, \mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(y)](c) + \mathcal{M}_{\mu_{\mathcal{A}}}[x, \mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(y)](a) = 0, \quad (3.58)$$

$$\mu_{\mathcal{B}}[\mathcal{R}_{\mu_{\mathcal{A}}}(x, y)(a), b, c] + \mu_{\mathcal{B}}[a, \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(b), c] + \mathcal{R}_{\mu_{\mathcal{A}}}[x, \mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(y)](a) = 0, \quad (3.59)$$

$$\mathcal{L}_{\mu_{\mathcal{A}}}[\mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(x), y](c) + \mu_{\mathcal{B}}[a, \mathcal{R}_{\mu_{\mathcal{A}}}(x, y)(b), c] + \mu_{\mathcal{B}}[a, b, \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(c)] = 0, \quad (3.60)$$

$$\mathcal{L}_{\mu_{\mathcal{A}}}[\mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(x), y](c) + \mu_{\mathcal{B}}[a, \mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(b), c] + \mathcal{R}_{\mu_{\mathcal{A}}}[x, \mathcal{M}_{\mu_{\mathcal{B}}}(b, c)(y)](a) = 0, \quad (3.61)$$

$$\mathcal{L}_{\mu_{\mathcal{A}}}[\mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(x), y](c) + \mathcal{L}_{\mu_{\mathcal{A}}}[x, \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(y)](c) + \mathcal{M}_{\mu_{\mathcal{A}}}[x, \mathcal{M}_{\mu_{\mathcal{B}}}(b, c)(y)](a) = 0, \quad (3.62)$$

$$\mathcal{M}_{\mu_{\mathcal{A}}}[\mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(x), y](c) + \mathcal{R}_{\mu_{\mathcal{A}}}[\mathcal{M}_{\mu_{\mathcal{B}}}(b, c)(x), y](a) + \mu_{\mathcal{B}}[a, b, \mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(c)] = 0, \quad (3.63)$$

$$\mathcal{M}_{\mu_{\mathcal{A}}}[\mathcal{M}_{\mu_{\mathcal{B}}}(a, b)(x), y](c) + \mathcal{R}_{\mu_{\mathcal{A}}}[\mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(x), y](a) + \mathcal{R}_{\mu_{\mathcal{A}}}[x, \mathcal{L}_{\mu_{\mathcal{B}}}(b, c)(y)](a) = 0, \quad (3.64)$$

$$\mathcal{M}_{\mu_{\mathcal{A}}}[\mathcal{R}_{\mu_{\mathcal{B}}}(a, b)(x), y](c) + \mathcal{M}_{\mu_{\mathcal{A}}}(x, y)(\mu_{\mathcal{B}}(a, b, c)) + \mathcal{M}_{\mu_{\mathcal{A}}}[x, \mathcal{L}_{\mu_{\mathcal{B}}}(b, c)(y)](a) = 0, \quad (3.65)$$

$$\mathcal{R}_{\mu_{\mathcal{A}}}(x, y)(\mu_{\mathcal{B}}(a, b, c)) + \mathcal{R}_{\mu_{\mathcal{A}}}[\mathcal{L}_{\mu_{\mathcal{B}}}(b, c)(x), y](a) + \mu_{\mathcal{B}}[a, b, \mathcal{R}_{\mu_{\mathcal{A}}}(x, y)(c)] = 0, \quad (3.66)$$

for any  $x, y, z \in \mathcal{A}, a, b, c \in \mathcal{B}$ . Then, there is an associative partially 3-ary algebra structure on the direct sum  $\mathcal{A} \oplus \mathcal{B}$  of the underlying vector spaces of  $\mathcal{A}$  and  $\mathcal{B}$  given by the product  $\tau$  defined by

$$\begin{aligned} \tau[(x + a), (y + b), (z + c)] &= [\mu_{\mathcal{A}}(x, y, z) + \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(z) + \mathcal{M}_{\mu_{\mathcal{B}}}(a, c)(y) + \mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(x)] + \\ &\quad [\mu_{\mathcal{B}}(a, b, c) + \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(c) + \mathcal{M}_{\mu_{\mathcal{A}}}(x, z)(b) + \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(a)] \end{aligned}$$

for any  $x, y, z \in \mathcal{A}, a, b, c \in \mathcal{B}$ . Let  $\mathcal{A} \bowtie_{\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}}^{\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}} \mathcal{B}$  denote this associative partially 3-ary algebra.

**Proof** Let  $x_1, x_2, x_3, x_4, x_5 \in \mathcal{A}$  and  $y_1, y_2, y_3, y_4, y_5 \in \mathcal{B}$ . By definition, we have

$$\begin{aligned} \tau[(x + a), (y + b), (z + c)] &= [\mu_{\mathcal{A}}(x, y, z) + \mathcal{L}_{\mu_{\mathcal{B}}}(a, b)(z) + \mathcal{M}_{\mu_{\mathcal{B}}}(a, c)(y) + \mathcal{R}_{\mu_{\mathcal{B}}}(b, c)(x)] + \\ &\quad [\mu_{\mathcal{B}}(a, b, c) + \mathcal{L}_{\mu_{\mathcal{A}}}(x, y)(c) + \mathcal{M}_{\mu_{\mathcal{A}}}(x, z)(b) + \mathcal{R}_{\mu_{\mathcal{A}}}(y, z)(a)] \end{aligned}$$

for any  $x, y, z \in \mathcal{A}, a, b, c \in \mathcal{B}$ . Setting the strong condition

$$\begin{aligned} &\tau[\tau[(x_1 + y_1), (x_2 + y_2), (x_3 + y_3)], (x_4 + y_4), (x_5 + y_5)] \\ &= \tau[(x_1 + y_1), \tau[(x_2 + y_2), (x_3 + y_3), (x_4 + y_4)], (x_5 + y_5)] \\ &= \tau[(x_1 + y_1), (x_2 + y_2), \tau[(x_3 + y_3), (x_4 + y_4), (x_5 + y_5)]], \end{aligned}$$

we obtain by direct computation the Eqs.(3.47) - (3.66). Then, there is an associative partially 3-ary algebra structure on the direct sum  $\mathcal{A} \oplus \mathcal{B}$  of the underlying vector spaces of  $\mathcal{A}$  and  $\mathcal{B}$  if and only if the Eqs.(3.47) - (3.66) are satisfied.  $\square$

DEFINITION 3.14. Let  $(\mathcal{A}, \mu_{\mathcal{A}})$  and  $(\mathcal{B}, \mu_{\mathcal{B}})$  be two partially associative 3-ary algebras. Suppose that there are linear maps  $\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}$  and  $\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}$  such that  $(\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{B})$  is a quasi trimodule of  $\mathcal{A}$ , and  $(\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}, \mathcal{A})$  is a quasi trimodule of  $\mathcal{B}$ . If Eqs.3.47 - 3.66 are satisfied, then  $(\mathcal{A}, \mathcal{B}, \mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}})$  is called a matched pair of partially associative 3-ary algebras.

DEFINITION 3.15. Let  $(\mathcal{A}, \mu_{\mathcal{A}})$  and  $(\mathcal{B}, \mu_{\mathcal{B}})$  be two partially associative 3-ary algebras. Suppose that there are linear maps  $\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}$  and  $\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}$  such that  $(\mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{B})$  is a trimodule of  $\mathcal{A}$ , and  $(\mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}}, \mathcal{A})$  is a trimodule of  $\mathcal{B}$ .

Then  $(\mathcal{A}, \mathcal{B}, \mathcal{L}_{\mu_{\mathcal{A}}}, \mathcal{M}_{\mu_{\mathcal{A}}}, \mathcal{R}_{\mu_{\mathcal{A}}}, \mathcal{L}_{\mu_{\mathcal{B}}}, \mathcal{M}_{\mu_{\mathcal{B}}}, \mathcal{R}_{\mu_{\mathcal{B}}})$  is a matched pair of partially associative 3-ary algebras if Eqs.3.47 - 3.66 and the following two conditions are satisfied:

$$\mathcal{M}_{\mu_{\mathcal{A}}}(a, z)(\mathcal{M}_{\mu_{\mathcal{A}}}(b, y)(\mathcal{M}_{\mu_{\mathcal{A}}}(c, x)(v'))) = \mathcal{M}_{\mu_{\mathcal{A}}}(\mu_{\mathcal{A}}(a, b, c), \mu_{\mathcal{A}}(x, y, z))(v'), \quad (3.67)$$

$$\mathcal{M}_{\mu_{\mathcal{B}}}(a', z')(\mathcal{M}_{\mu_{\mathcal{B}}}(b', y')(\mathcal{M}_{\mu_{\mathcal{B}}}(c', x')(v))) = \mathcal{M}_{\mu_{\mathcal{B}}}(\mu_{\mathcal{B}}(a', b', c'), \mu_{\mathcal{B}}(x', y', z'))(v), \quad (3.68)$$

for any  $x, y, z, a, b, c, v \in \mathcal{A}, x', y', z', a', b', c', v' \in \mathcal{B}$ .

LEMMA 3.16. Let  $(\mathcal{L}_{\mu}, \mathcal{M}_{\mu}, \mathcal{R}_{\mu})$  be a trimodule of a partially associative 3-ary algebra  $\mathcal{A}$ . Then, the linear maps  $\mathcal{L}_{\mu}^*, \mathcal{M}_{\mu}^*, \mathcal{R}_{\mu}^* : \mathcal{A} \otimes \mathcal{A} \rightarrow gl(V^*)$  given by

$$\begin{aligned} \langle \mathcal{L}_{\mu}^*(x, y)u^*, v \rangle &= \langle \mathcal{L}_{\mu}(x, y)v, u^* \rangle; \langle \mathcal{M}_{\mu}^*(x, y)u^*, v \rangle = \langle \mathcal{M}_{\mu}(x, y)v, u^* \rangle; \\ \langle \mathcal{R}_{\mu}^*(x, y)u^*, v \rangle &= \langle \mathcal{R}_{\mu}(x, y)v, u^* \rangle; \text{ for all } x, y \in \mathcal{A}, v \in V, u^* \in V^*. \end{aligned}$$

realize a trimodule of  $\mathcal{A}$  denoted by  $(\mathcal{R}_{\mu}^*, \mathcal{M}_{\mu}^*, \mathcal{L}_{\mu}^*)$  satisfying

$$\mathcal{R}_{\mu}^*(a, b)(\mathcal{R}_{\mu}^*(c, d)(u^*)) + \mathcal{R}_{\mu}^*(\mu(a, b, c), d)(u^*) + \mathcal{R}_{\mu}^*(a, \mu(b, c, d))(u^*) = 0, \quad (3.69)$$

$$\mathcal{L}_{\mu}^*(c, d)(\mathcal{L}_{\mu}^*(a, b)(u^*)) + \mathcal{L}_{\mu}^*(a, \mu(b, c, d))(u^*) + \mathcal{L}_{\mu}^*(\mu(a, b, c), d)(u^*) = 0, \quad (3.70)$$

$$\mathcal{M}_{\mu}^*(a, z)(\mathcal{M}_{\mu}^*(b, y)(\mathcal{M}_{\mu}^*(c, x)(u^*))) = \mathcal{M}_{\mu}^*(\mu(c, b, a), \mu(z, y, x))(u^*), \quad (3.71)$$

$$\mathcal{M}_{\mu}^*(a, d)(\mathcal{R}_{\mu}^*(b, c)(u^*)) + \mathcal{R}_{\mu}^*(d, b)(\mathcal{M}_{\mu}^*(a, c)(u^*)) + \mathcal{M}_{\mu}^*(a, \mu(d, b, c))(u^*) = 0, \quad (3.72)$$

$$\mathcal{M}_{\mu}^*(a, d)(\mathcal{R}_{\mu}^*(b, c)(u^*)) + \mathcal{L}_{\mu}^*(c, a)(\mathcal{M}_{\mu}^*(b, d)(u^*)) + \mathcal{M}_{\mu}^*(\mu(b, c, a), d)(u^*) = 0, \quad (3.73)$$

$$\mathcal{L}_{\mu}^*(c, d)(\mathcal{R}_{\mu}^*(a, b)(u^*)) + \mathcal{R}_{\mu}^*(a, b)(\mathcal{L}_{\mu}^*(c, d)(u^*)) + \mathcal{M}_{\mu}^*(d, a)(\mathcal{M}_{\mu}^*(c, b)(u^*)) = 0, \quad (3.74)$$

$\forall a, b, c, d, x, y, z \in \mathcal{A}, u^* \in V^*$ .

PROPOSITION 3.17. Let  $(\mathcal{A}, \mu)$  be a partially associative 3-ary algebra. Then,  $(L_{\mu}, 0, 0, \mathcal{A})$ ,  $(0, 0, R_{\mu}, \mathcal{A})$ ,  $(L_{\mu}, M_{\mu}, R_{\mu}, \mathcal{A})$ ,  $(R_{\mu}^*, 0, 0, \mathcal{A}^*)$ ,  $(0, 0, L_{\mu}^*, \mathcal{A}^*)$  and  $(R_{\mu}^*, M_{\mu}^*, L_{\mu}^*, \mathcal{A}^*)$  are trimodules of  $\mathcal{A}$ .

THEOREM 3.18. Let  $(\mathcal{A}, \mu)$  be a partially associative 3-ary algebra. Suppose that there is a partially associative 3-ary algebra structure  $\nu$  on its dual space  $\mathcal{A}^*$ .

Then,  $(\mathcal{A}, \mathcal{A}^*, R_{\mu}^*, M_{\mu}^*, L_{\mu}^*, R_{\nu}^*, M_{\nu}^*, L_{\nu}^*)$  is a matched pair of partially associative 3-ary algebras if and only if there is an associative partially 3-ary algebra structure on the direct sum  $\mathcal{A} \oplus \mathcal{A}^*$  of the underlying vector spaces of  $\mathcal{A}$  and  $\mathcal{A}^*$  given by the product  $\tau$  defined by

$$\begin{aligned} \tau[(x + a^*), (y + b^*), (z + c^*)] &= [\mu(x, y, z) + L_{\nu}^*(a^*, b^*)(z) + M_{\nu}^*(a^*, c^*)(y) + R_{\nu}^*(b^*, c^*)(x)] + \\ &\quad [\nu(a^*, b^*, c^*) + L_{\mu}^*(x, y)(c^*) + M_{\mu}^*(x, z)(b^*) + R_{\mu}^*(y, z)(a^*)] \end{aligned}$$

for any  $x, y, z \in \mathcal{A}, a^*, b^*, c^* \in \mathcal{A}^*$ .

#### 4. Associative 3-ary infinitesimal bialgebras

We start this section by introducing the notion of associative 3-ary infinitesimal bialgebras, which is characterized by a compatible condition between the 3-ary product and the 3-ary co-product. Let us also mention the work by [13] on ternary Hopf algebras where a construction of ternary bialgebra is given.

#### 4.1. Definitions.

DEFINITION 4.1. A partially associative 3-ary infinitesimal bialgebra is a triple  $(\mathcal{A}, \mu, \Delta)$  such that

- (1)  $(\mathcal{A}, \mu)$  is a partially associative 3-ary algebra,
- (2)  $(\mathcal{A}, \Delta)$  is a partially coassociative 3-ary coalgebra,
- (3)  $\Delta$  and  $\mu$  satisfy the following condition

$$\Delta\mu(x, y, z) = (L_\mu(x, y) \otimes \text{id} \otimes \text{id})\Delta(z) + (\text{id} \otimes M_\mu(x, z) \otimes \text{id})\Delta(y) + (\text{id} \otimes \text{id} \otimes R_\mu(y, z))\Delta(x). \quad (4.1)$$

EXAMPLE 4.2. Let  $\mathcal{P}$  be a 2-dimensionnal vector space with a basis  $\{e_1, e_2\}$ . Consider the following products defined by

$$\begin{aligned} \mu : \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P} &\rightarrow \mathcal{P}, \quad \mu(e_1 \otimes e_1 \otimes e_1) = e_2, \mu(e_i \otimes e_j \otimes e_k) = 0, i, j, k = 1, 2, (i, j, k) \neq (1, 1, 1); \\ \Delta : \mathcal{P} &\rightarrow \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P}, \quad \Delta(e_1) = e_2 \otimes e_2 \otimes e_2, \quad \Delta(e_2) = 0. \end{aligned}$$

By a direct computation, we obtain that the triple  $(\mathcal{P}, \mu, \Delta)$  is a partially associative 3-ary infinitesimal bialgebra.

DEFINITION 4.3. A totally associative 3-ary infinitesimal bialgebra is a triple  $(\mathcal{A}, \mu, \Delta)$  such that

- (1)  $(\mathcal{A}, \mu)$  is a totally associative 3-ary algebra,
- (2)  $(\mathcal{A}, \Delta)$  is a totally coassociative 3-ary coalgebra,
- (3)  $\Delta$  and  $\mu$  satisfy the condition (4.1).

EXAMPLE 4.4. Let  $\mathcal{T}$  be a 2-dimensionnal vector space with a basis  $\{e_1, e_2\}$ . Consider the following products defined by

$$\begin{aligned} \mu : \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} &\rightarrow \mathcal{T}, \\ \mu(e_1 \otimes e_1 \otimes e_1) &= e_1 & \mu(e_2 \otimes e_2 \otimes e_1) &= e_1 + e_2 \\ \mu(e_1 \otimes e_1 \otimes e_2) &= e_2 & \mu(e_2 \otimes e_2 \otimes e_2) &= e_1 + 2e_2 \\ \mu(e_1 \otimes e_2 \otimes e_1) &= e_2 & \mu(e_1 \otimes e_2 \otimes e_2) &= e_1 + e_2 \\ \mu(e_2 \otimes e_1 \otimes e_1) &= e_2 & \mu(e_2 \otimes e_1 \otimes e_2) &= e_1 + e_2 \end{aligned}$$

and

$$\begin{aligned} \Delta : \mathcal{T} &\rightarrow \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T}, \\ \Delta(e_1) &= e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 \\ \Delta(e_2) &= e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \\ &\quad + e_2 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + 2e_2 \otimes e_2 \otimes e_2. \end{aligned}$$

By a direct computation, we prove that the triple  $(\mathcal{T}, \mu, \Delta)$  is a totally associative 3-ary infinitesimal bialgebra.

DEFINITION 4.5. A weak totally associative 3-ary infinitesimal bialgebra is a triple  $(\mathcal{A}, \mu, \Delta)$  such that

- (1)  $(\mathcal{A}, \mu)$  is a weak totally associative 3-ary algebra,
- (2)  $(\mathcal{A}, \Delta)$  is a weak totally coassociative 3-ary coalgebra,
- (3)  $\Delta$  and  $\mu$  satisfy the condition (4.1).

DEFINITION 4.6. Two associative 3-ary infinitesimal bialgebras  $(\mathcal{A}_1, \mu_1, \Delta_1)$  and  $(\mathcal{A}_2, \mu_2, \Delta_2)$  are called equivalent if there exists a vector space isomorphism  $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  such that

- (1)  $f : (\mathcal{A}_1, \mu_1) \rightarrow (\mathcal{A}_2, \mu_2)$  is an associative 3-ary algebra isomorphism, that is,

$$f\mu_1(x, y, z) = \mu_2(f(x), f(y), f(z)) \text{ for all } x, y, z \in \mathcal{A}_1;$$

- (2)  $f : (\mathcal{A}_1, \Delta_1) \rightarrow (\mathcal{A}_2, \Delta_2)$  is a coassociative 3-ary coalgebra isomorphism, that is,

$$\Delta_2(f(x)) = (f \otimes f \otimes f)\Delta_1(x) \text{ for every } x \in \mathcal{A}_1.$$

EXAMPLE 4.7. Let  $\mathcal{P}$  be a 2-dimensionnal vector space with a basis  $\{e_1, e_2\}$ . Consider the following products  $\mu_1, \mu_2 : \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P}$ ,  $\Delta_1, \Delta_2 : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P}$ , defined by

$$\begin{aligned} \mu_1(e_1 \otimes e_1 \otimes e_1) &= e_2, \mu_1(e_i \otimes e_j \otimes e_k) = 0, i, j, k = 1, 2, (i, j, k) \neq (1, 1, 1); \\ \Delta_1(e_1) &= e_2 \otimes e_2 \otimes e_2, \Delta_1(e_2) = 0. \end{aligned}$$

and

$$\begin{aligned} \mu_2(e_2 \otimes e_2 \otimes e_2) &= e_1, \mu_2(e_i \otimes e_j \otimes e_k) = 0, i, j, k = 1, 2, (i, j, k) \neq (2, 2, 2); \\ \Delta_2(e_2) &= e_1 \otimes e_1 \otimes e_1, \Delta_2(e_1) = 0. \end{aligned}$$

By a direct computation, we show that the triples  $(\mathcal{P}, \mu_1, \Delta_1)$  and  $(\mathcal{P}, \mu_2, \Delta_2)$  are partially associative 3-ary infinitesimal bialgebras.

Let us consider the linear map  $f : \mathcal{P} \rightarrow \mathcal{P}$ ,  $f(e_1) = e_2, f(e_2) = e_1$ . Then,  $(\mathcal{P}, \mu_1, \Delta_1)$  and  $(\mathcal{P}, \mu_2, \Delta_2)$  are equivalent partially associative 3-ary infinitesimal bialgebras by the isomorphism  $f : (\mathcal{P}, \mu_1, \Delta_1) \rightarrow (\mathcal{P}, \mu_2, \Delta_2)$ .

#### 4.2. Main results.

THEOREM 4.8. Let  $(\mathcal{P}, \mu, \Delta)$  be a finite dimensional partially associative 3-ary infinitesimal bialgebra. Then,  $(\mathcal{P}^*, \Delta^*, \mu^*)$  is a partially associative 3-ary infinitesimal bialgebra.

**Proof:**  $(\mathcal{P}, \mu, \Delta)$  is a finite dimensional partially associative 3-ary infinitesimal bialgebra. Then by the Theorem(2.9) and the Theorem(2.10),  $(\mathcal{P}^*, \Delta^*)$  is a partially associative 3-ary algebra in the multiplication (2.15), and  $(\mathcal{P}^*, \mu^*)$  is a partially coassociative 3-ary coalgebra in the multiplication (2.4). Now, we prove that  $\mu^* : \mathcal{P}^* \rightarrow \mathcal{P}^* \otimes \mathcal{P}^* \otimes \mathcal{P}^*$  satisfies the identity (4.1), that is, the following identity holds for every  $\xi, \eta, \gamma \in \mathcal{P}^*$ ,

$$\begin{aligned} \mu^*(\Delta^*(\xi, \eta, \gamma)) &= (L_{\Delta^*}(\xi, \eta) \otimes \text{id}^* \otimes \text{id}^*)\mu^*(\gamma) + (\text{id}^* \otimes M_{\Delta^*}(\xi, \gamma) \otimes \text{id}^*)\mu^*(\eta) \\ &\quad + (\text{id}^* \otimes \text{id}^* \otimes R_{\Delta^*}(\eta, \gamma))\mu^*(\xi) \end{aligned} \quad (4.2)$$

For every  $x, y, z \in \mathcal{P}$  and  $\xi, \eta, \gamma \in \mathcal{P}^*$ , by identities (2.15) and (2.4),

$$\begin{aligned} \langle \mu^* \Delta^*(\xi, \eta, \gamma), x \otimes y \otimes z \rangle &= \langle \xi \otimes \eta \otimes \gamma, \Delta \mu(x \otimes y \otimes z) \rangle \\ &= \langle \xi \otimes \eta \otimes \gamma, (L_\mu(x, y) \otimes \text{id} \otimes \text{id})\Delta(z) + \\ &\quad (\text{id} \otimes M_\mu(x, z) \otimes \text{id})\Delta(y) + \\ &\quad (\text{id} \otimes \text{id} \otimes R_\mu(y, z))\Delta(x) \rangle \\ &= \langle \xi \otimes \eta \otimes \gamma, (L_\mu(x, y) \otimes \text{id} \otimes \text{id})\Delta(z) \rangle + \\ &\quad \langle \xi \otimes \eta \otimes \gamma, (\text{id} \otimes M_\mu(x, z) \otimes \text{id})\Delta(y) \rangle + \\ &\quad \langle \xi \otimes \eta \otimes \gamma, (\text{id} \otimes \text{id} \otimes R_\mu(y, z))\Delta(x) \rangle \\ &= \langle \Delta^*(\text{id}^* \otimes \text{id}^* \otimes L_\mu^*(x, y))(\xi \otimes \eta \otimes \gamma), z \rangle + \\ &\quad \langle \Delta^*(\text{id}^* \otimes M_\mu^*(x, z) \otimes \text{id}^*)(\xi \otimes \eta \otimes \gamma), y \rangle + \\ &\quad \langle \Delta^*(R_\mu^*(y, z) \otimes \text{id}^* \otimes \text{id}^*)(\xi \otimes \eta \otimes \gamma), x \rangle \\ &= \langle \Delta^*(\xi \otimes \eta \otimes L_\mu^*(x, y)(\gamma)), z \rangle + \\ &\quad \langle \Delta^*(\xi \otimes M_\mu^*(x, z)(\eta) \otimes \gamma), y \rangle + \\ &\quad \langle \Delta^*(R_\mu^*(y, z)(\xi) \otimes \eta \otimes \gamma), x \rangle \\ &= \langle L_{\Delta^*}(\xi, \eta)(L_\mu^*(x, y)(\gamma)), z \rangle + \\ &\quad \langle M_{\Delta^*}(\xi, \gamma)(M_\mu^*(x, z)(\eta)), y \rangle + \\ &\quad \langle R_{\Delta^*}(\eta, \gamma)(R_\mu^*(y, z)(\xi)), x \rangle \\ &= \langle L_\mu^*(x, y)(\gamma), L_{\Delta^*}^*(\xi, \eta)(z) \rangle + \\ &\quad \langle M_\mu^*(x, z)(\eta), M_{\Delta^*}^*(\xi, \gamma)(y) \rangle + \\ &\quad \langle R_\mu^*(y, z)(\xi), R_{\Delta^*}^*(\eta, \gamma)(x) \rangle \\ &= \langle \gamma, \mu(x, y, L_{\Delta^*}^*(\xi, \eta)(z)) \rangle + \\ &\quad \langle \eta, \mu(x, M_{\Delta^*}^*(\xi, \gamma)(y), z) \rangle + \\ &\quad \langle \gamma, \mu(R_{\Delta^*}^*(\eta, \gamma)(x), y, z) \rangle \\ &= \langle \mu^*(\gamma), x \otimes y \otimes L_{\Delta^*}^*(\xi, \eta)(z) \rangle + \\ &\quad \langle \mu^*(\eta), x \otimes M_{\Delta^*}^*(\xi, \gamma)(y) \otimes z \rangle + \\ &\quad \langle \mu^*(\gamma), R_{\Delta^*}^*(\eta, \gamma)(x) \otimes y \otimes z \rangle \\ &= \langle \mu^*(\gamma), (\text{id} \otimes \text{id} \otimes L_{\Delta^*}^*(\xi, \eta))(x \otimes y \otimes z) \rangle + \end{aligned}$$

$$\begin{aligned}
& \langle \mu^*(\eta), (\text{id} \otimes M_{\Delta^*}(\xi, \gamma) \otimes \text{id})(x \otimes y \otimes z) \rangle + \\
& \langle \mu^*(\xi), (R_{\Delta^*}(\eta, \gamma) \otimes \text{id} \otimes \text{id})(x \otimes y \otimes z) \rangle \\
= & \langle (L_{\Delta^*}(\xi, \eta) \otimes \text{id}^* \otimes \text{id}^*)\mu^*(\gamma), x \otimes y \otimes z \rangle + \\
& \langle (\text{id}^* \otimes M_{\Delta^*}(\xi, \gamma) \otimes \text{id}^*)\mu^*(\eta), x \otimes y \otimes z \rangle + \\
& \langle \text{id}^* \otimes \text{id}^* \otimes (R_{\Delta^*}(\eta, \gamma))\mu^*(\xi), x \otimes y \otimes z \rangle.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\mu^* \Delta^*(\xi, \eta, \gamma) = & (\text{id}^* \otimes \text{id}^* \otimes R_{\Delta^*}(\eta, \gamma))\mu^*(\xi) + (\text{id}^* \otimes M_{\Delta^*}(\xi, \gamma) \otimes \text{id}^*)\mu^*(\eta) + \\
& (L_{\Delta^*}(\xi, \eta) \otimes \text{id}^* \otimes \text{id}^*)\mu^*(\gamma).
\end{aligned}$$

Hence the identity (4.2) holds.  $\square$

The triple  $(\mathcal{P}^*, \Delta^*, \mu^*)$  is called the dual partially associative 3-ary infinitesimal bialgebra of  $(\mathcal{P}, \mu, \Delta)$ .

EXAMPLE 4.9. From Theorem (4.8), the dual partially associative 3-ary infinitesimal bialgebra  $(\mathcal{P}^*, \Delta^*, \mu^*)$  of  $(\mathcal{P}, \mu, \Delta)$  in Example (4.2) has its multiplications  $\Delta^* : \mathcal{P}^* \otimes \mathcal{P}^* \otimes \mathcal{P}^* \rightarrow \mathcal{P}^*$  and  $\mu^* : \mathcal{P}^* \rightarrow \mathcal{P}^* \otimes \mathcal{P}^* \otimes \mathcal{P}^*$ , defined by:

$$\begin{aligned}
\Delta^*(e_2^* \otimes e_2^* \otimes e_2^*) &= e_1^*, \Delta^*(e_i^* \otimes e_j^* \otimes e_k^*) = 0, i, j, k = 1, 2, (i, j, k) \neq (2, 2, 2); \\
\mu^*(e_2^*) &= e_1^* \otimes e_1^* \otimes e_1^*, \mu^*(e_1^*) = 0.
\end{aligned}$$

Similarly, we have the following results.

THEOREM 4.10. Let  $(\mathcal{A}, \mu, \Delta)$  be a finite dimensional totally associative 3-ary infinitesimal bialgebra. Then,  $(\mathcal{A}^*, \Delta^*, \mu^*)$  is a totally associative 3-ary infinitesimal bialgebra.

The triple  $(\mathcal{A}^*, \Delta^*, \mu^*)$  is called the dual totally associative 3-ary infinitesimal bialgebra of  $(\mathcal{A}, \mu, \Delta)$ .

EXAMPLE 4.11. From Theorem (4.10), the dual totally associative 3-ary infinitesimal bialgebra  $(\mathcal{T}^*, \Delta^*, \mu^*)$  of  $(\mathcal{T}, \mu, \Delta)$  in Example (4.4) has its multiplications  $\Delta^* : \mathcal{T}^* \otimes \mathcal{T}^* \otimes \mathcal{T}^* \rightarrow \mathcal{T}^*$  and  $\mu^* : \mathcal{T}^* \rightarrow \mathcal{T}^* \otimes \mathcal{T}^* \otimes \mathcal{T}^*$ , defined such that  $\Delta^* = \mu$  and  $\mu^* = \Delta$ .

THEOREM 4.12. Let  $(\mathcal{A}, \mu, \Delta)$  be a finite dimensional weak totally associative 3-ary infinitesimal bialgebra. Then,  $(\mathcal{A}^*, \Delta^*, \mu^*)$  is a weak totally associative 3-ary bialgebra.

The triple  $(\mathcal{A}^*, \Delta^*, \mu^*)$  is called the dual weak totally associative 3-ary infinitesimal bialgebra of  $(\mathcal{A}, \mu, \Delta)$ .

Now, we study 3-ary associative bialgebras by means of structure constants.

Let  $(\mathcal{A}, \mu, \Delta)$  be a partially 3-ary associative infinitesimal bialgebra with the multiplications in the basis  $e_1, \dots, e_n$  as follows:

$$\mu(e_i, e_j, e_k) = \sum_{l=1}^n c_{ijk}^l e_l, \quad \Delta(e_l) = \sum_{1 \leq r, s, t \leq n} a_l^{rst} e_r \otimes e_s \otimes e_t, \quad (4.3)$$

where  $c_{ijk}^l, a_l^{rst} \in \mathcal{K}, 1 \leq i, j, k, l \leq n$ . Then,

$$\Delta\mu(e_i, e_j, e_k) = \Delta\left(\sum_{l=1}^n c_{ijk}^l e_l\right) = \sum_{l=1}^n c_{ijk}^l \Delta(e_l) = \sum_{l=1}^n \sum_{1 \leq r, s, t \leq n} c_{ijk}^l a_l^{rst} e_r \otimes e_s \otimes e_t \quad (4.4)$$

From the identity (4.1), we have:

$$\begin{aligned}
\Delta\mu(e_i, e_j, e_k) &= (L_\mu(e_i, e_j) \otimes \text{id} \otimes \text{id})\Delta(e_k) + (\text{id} \otimes M_\mu(e_i, e_k) \otimes \text{id})\Delta(e_j) + \\
&\quad (\text{id} \otimes \text{id} \otimes R_\mu(e_j, e_k))\Delta(e_i) \\
&= (L_\mu(e_i, e_j) \otimes \text{id} \otimes \text{id}) \left( \sum_{1 \leq r, s, t \leq n} a_k^{rst} e_r \otimes e_s \otimes e_t \right) + \\
&\quad (\text{id} \otimes M_\mu(e_i, e_k) \otimes \text{id}) \left( \sum_{1 \leq r, s, t \leq n} a_j^{rst} e_r \otimes e_s \otimes e_t \right) +
\end{aligned}$$



$$\begin{aligned}
& (\text{id} \otimes \text{id} \otimes R_\mu(e_j, e_k)) \left( \sum_{1 \leq r, s, t \leq n} a_i^{rst} e_r \otimes e_s \otimes e_t \right) \\
&= \sum_{1 \leq r, s, t \leq n} [a_k^{rst} \mu(e_i, e_j, e_r) \otimes e_s \otimes e_t + a_j^{rst} e_r \otimes \mu(e_i, e_s, e_k) \otimes e_t + \\
&\quad a_i^{rst} e_r \otimes e_s \otimes \mu(e_t, e_j, e_k)] \\
&= \sum_{1 \leq r, s, t \leq n} \sum_{l=1}^n [a_k^{rst} c_{ijr}^l e_l \otimes e_s \otimes e_t + a_j^{rst} c_{isk}^l e_r \otimes e_l \otimes e_t + a_i^{rst} c_{tjk}^l e_r \otimes e_s \otimes e_l] \\
&= \sum_{1 \leq r, s, t \leq n} \sum_{l=1}^n [a_k^{lst} c_{ijl}^r + a_j^{rlt} c_{ilk}^s + a_i^{rsl} c_{ljk}^t] e_r \otimes e_s \otimes e_t. \tag{4.5}
\end{aligned}$$

Comparing the identities (4.4) and (4.5), we infer

$$\sum_{l=1}^n [a_k^{lst} c_{ijl}^r + a_j^{rlt} c_{ilk}^s + a_i^{rsl} c_{ljk}^t - a_l^{rst} c_{ijk}^l] = 0. \tag{4.6}$$

Conversely, if an associative 3-ary algebra  $(\mathcal{A}, \mu)$ , and a coassociative 3-ary coalgebra defined by (4.3) satisfy the identity (4.5), then  $\mu$  and  $\Delta$  satisfy the identity (4.1). Therefore, we have the following result:

**THEOREM 4.13.** *Let  $\mathcal{A}$  be a vector space with a basis  $e_1, \dots, e_n$ , and  $(\mathcal{A}, \mu)$  and  $(\mathcal{A}, \Delta)$  be associative 3-ary algebra and coassociative 3-ary coalgebra, respectively, defined by (4.3). Then,  $(\mathcal{A}, \mu, \Delta)$  is an associative 3-ary infinitesimal bialgebra, if and only if  $c_{ijk}^l$  and  $a_l^{rst}$ ,  $1 \leq i, j, k, l, r, s, t \leq n$ , satisfy identity (4.5).*

Let  $\mathcal{A}$  be an associative 3-ary algebra. Considering the exchange operator  $\sigma : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  defined as

$$\sigma(a \otimes b) = b \otimes a, \text{ for all } a, b \in \mathcal{A},$$

the condition (4.1) can be rewritten as:

$$\begin{aligned}
\Delta\mu &= (\mu \otimes \text{id} \otimes \text{id})(\text{id} \otimes \text{id} \otimes \Delta) + (\text{id} \otimes \mu \otimes \text{id})(\sigma \otimes \text{id} \otimes \sigma)(\text{id} \otimes \Delta \otimes \text{id}) \\
&\quad + (\text{id} \otimes \text{id} \otimes \mu)(\Delta \otimes \text{id} \otimes \text{id}). \tag{4.7}
\end{aligned}$$

The relation (4.7) can be re-expressed in terms of elements as follows:

$$\Delta\mu(a, b, c) = \sum_{(c)} abc_1 \otimes c_2 \otimes c_3 + \sum_{(b)} b_1 \otimes ab_2c \otimes b_3 + \sum_{(a)} a_1 \otimes a_2 \otimes a_3bc, \text{ for all } a, b, c \in \mathcal{A}, \tag{4.8}$$

or, equivalently,

$$\Delta\mu(a, b, c) = (L_\mu(a, b) \otimes \text{id} \otimes \text{id})\Delta(c) + (\text{id} \otimes M_\mu(a, c) \otimes \text{id})\Delta(b) + (\text{id} \otimes \text{id} \otimes R_\mu(b, c))\Delta(a). \tag{4.9}$$

**DEFINITION 4.14.** *An associative 3-ary infinitesimal bialgebra is a triple  $(\mathcal{A}, \mu, \Delta)$  consisting of an associative 3-ary algebra  $(\mathcal{A}, \mu)$  and a coassociative 3-ary coalgebra  $(\mathcal{A}, \Delta)$  such that the condition (4.7) holds.*

**PROPOSITION 4.15.** *Let  $(\mathcal{A}, \mu, \Delta)$  be an associative 3-ary infinitesimal bialgebra. Then so are  $(\mathcal{A}, -\mu, \Delta)$ ,  $(\mathcal{A}, \mu, -\Delta)$ ,  $(\mathcal{A}, -\mu, -\Delta)$ .*

**Proof:** Using the Eq.(4.7), we obtain the results by a direct computation. □

**THEOREM 4.16.** *Let  $(\mathcal{A}, \mu, \Delta)$  be a finite dimensional associative 3-ary infinitesimal bialgebra. Then so is  $(\mathcal{A}^*, \Delta^*, \mu^*)$ .*

**Proof:** One has checked that  $(\mathcal{A}^*, \Delta^*)$  is an associative 3-ary algebra and  $(\mathcal{A}^*, \mu^*)$  is a coassociative 3-ary coalgebra. It remains to establish the condition (4.7) for  $\Delta^*$  and  $\mu^*$ . For that, let us consider  $a, b, c \in \mathcal{A}$  and  $\varphi, \psi, \chi \in \mathcal{A}^*$ . We obtain:

$$\langle \mu^* \circ \Delta^*(\varphi \otimes \psi \otimes \chi), a \otimes b \otimes c \rangle = \langle \varphi \otimes \psi \otimes \chi, \Delta \circ \mu(a \otimes b \otimes c) \rangle$$

$$\begin{aligned}
&= \langle \varphi \otimes \psi \otimes \chi, (\mu \otimes \text{id} \otimes \text{id})(\text{id} \otimes \text{id} \otimes \Delta)(a \otimes b \otimes c) \rangle + \\
&\quad \langle \varphi \otimes \psi \otimes \chi, (\text{id} \otimes \mu \otimes \text{id})(\sigma \otimes \text{id} \otimes \sigma)(\text{id} \otimes \Delta \otimes \text{id})(a \otimes b \otimes c) \rangle \\
&\quad + \langle \varphi \otimes \psi \otimes \chi, (\text{id} \otimes \text{id} \otimes \mu)(\Delta \otimes \text{id} \otimes \text{id})(a \otimes b \otimes c) \rangle \\
&= \langle (\text{id}^* \otimes \text{id}^* \otimes \Delta^*)(\mu^* \otimes \text{id}^* \otimes \text{id}^*)(\varphi \otimes \psi \otimes \chi), a \otimes b \otimes c \rangle + \\
&\quad \langle (\text{id}^* \otimes \Delta^* \otimes \text{id}^*)(\sigma^* \otimes \text{id}^* \otimes \sigma^*)(\text{id}^* \otimes \mu^* \otimes \text{id}^*)(\varphi \otimes \psi \otimes \chi), \\
&\quad a \otimes b \otimes c \rangle + \langle (\Delta^* \otimes \text{id}^* \otimes \text{id}^*)(\text{id}^* \otimes \text{id}^* \otimes \mu^*)(\varphi \otimes \psi \otimes \chi), \\
&\quad a \otimes b \otimes c \rangle \text{ where } \sigma^*(\varphi \otimes \psi) = \psi \otimes \varphi,
\end{aligned}$$

showing that the condition (4.7) holds for  $\Delta^*$  and  $\mu^*$ .  $\square$

## 5. Concluding remarks

The main results obtained in this study can be summarized as follows:

- i) Partially and totally coassociative 3-ary coalgebras are formulated in Definition 2.4, Definition 2.5 and Definition 2.6, and characterized in terms of the algebra constant structures in Theorem 2.7). The relation between a coassociative 3-ary coalgebra and its dual is given in Theorem 2.9).
- ii) Theorem 2.10 connects associative 3-ary algebras to coassociative 3-ary coalgebras.
- iii) Theorem 2.14 links the isomorphism between two coassociative 3-ary coalgebras to the isomorphism of their duals.
- iv) The definition of the trimodule of partially associative 3-ary algebras is given in Definition 3.10. The matched pairs of totally and partially associative 3-ary algebras are built in Theorem 3.4 and Theorem 3.13, respectively.
- v) Partially and totally associative 3-ary infinitesimal bialgebras are defined in Definition 4.1 and Definition 4.3, respectively, while their dual structures are provided in Theorem 4.8 and Theorem 4.10.

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